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Abstract

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MATHEMATICS

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ESTIMATES OF SOLUTIONS OF BOUNDARY-VALUE PROBLEMS WITH THE BESSEL OPERATOR

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In the modern theory of boundary-value problems, an important place is occupied by the question of limiting sharp estimates of the norms of solutions of boundary-value problems in terms of the norms of the right-hand sides and boundary conditions. The present note is devoted mainly to estimates of solutions of boundary-value problems for certain singular differential equations. These estimates can be obtained rather simply, once explicit formulas are known for the solutions of equations with constant coefficients.

In the metrics $L_{2,k}$, estimates of solutions were obtained earlier by I. A. Kipriyanov in the paper (^{1}).

1. Let E_{n+2}^{++} be the Euclidean space of points (x, y, t), $x = (x_1, \dots, x_n)$, $y \geq 0$, $t \geq 0$. In this domain we shall consider a solution $u(x, y, t)$ of the B-elliptic (^{1}) equation of order $2m$ with constant (complex) coefficients

$$\mathcal{L}(D, B_y)u = \sum_{i+2r=2m} a_{i,r} D^i B_y^r u = f(x, y, t) \quad \text{for } t > 0, \tag{1}$$

satisfying m boundary conditions

$$H_j(D, B_y)u = \sum b_{i,r}^j D^i B_y^r u = g_j(x, y) \quad \text{for } t = 0. \tag{2}$$

These conditions are expressed by differential operators H_j with constant coefficients. The order of the operator H_j is equal to m_j. The numbers m_j are nonnegative and may exceed the number $2m$—the order of the operator $\mathcal{L}$. In addition, $\mathcal{L}$ and all H_j are homogeneous. Here

$$B_y = -\left(\frac{\partial^2}{\partial y^2} + \frac{k}{y}\frac{\partial}{\partial y}\right), \quad k > 0,$$

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$$D = (D_x, D_t), \quad \text{where} \quad D_t = \frac{1}{i}\frac{\partial}{\partial t}.$$

The functions f and g_j for $t \geq 0$ and $y \geq 0$ are infinitely differentiable and have compact support. In what follows we shall assume that the boundary operators are connected with the operator $\mathcal{L}$ by the Ya. B. Lopatinskii condition (the complementing condition (^{2})).

For sufficiently large N, let C_B^N denote the class of functions continuously differentiable N times with respect to x and t, and continuously admitting $N/2$ applications of the Bessel operator with respect to the variable y in the case of even N, and $\frac{\partial}{\partial y} B_y^{(N-1)/2}$ in the case of odd N. Extend the function f to the entire half-space $y \geq 0$ so that $f_N \in C_B^N$. The solution of equation (1) with right-hand side f_N has (^{3}) the form

$$v = v_N(x, y, t) = \mathcal{E} * f_N = \int T_y^z \mathcal{E}(x - s, y, t - \tau) f_N(s, z, \tau) z^k \, ds \, dz \, d\tau, \tag{3}$$

where $\mathcal{E}$ is the fundamental solution of the B-elliptic equation, determined by the formula (^{3})

$$\mathcal{E}(P) = |P|^{2m-n-k-2}\Omega\left(\frac{P}{|P|}\right) + q(P)\ln|P|, \quad P = (x, y, t),$$

and T_y^z is the generalized shift operator

$$T_y^z f(y) = \frac{\Gamma((k+1)/2)}{\Gamma(1/2)\Gamma(k/2)} \int_0^\pi f\left(\sqrt{y^2 + z^2 - 2yz\cos\alpha}\right) \sin^{k-1}\alpha \, d\alpha. \tag{4}$$

Integration in formula (3) is carried out over the whole half-space $z > 0$.

Let $u = v_N + w$. To determine the function w we obtain the following problem: $\mathcal{L}w = 0$ for $t > 0$ and $H_{jw} = \varphi_j(x, y)$ for $t = 0$, where the function

$$\varphi_j(x, y) = g_j(x, y) - H_j v_N\big|_{t=0} \tag{5}$$

is no longer finite. Using the results of paper (^{4}), for the solution of the latter problem we obtain the representation

$$D^i B_y^r w = \sum_{j=1}^m \int_{E_{n+1}^+} D^i B_y^r T_y^z K_j(x - s, y, t) \varphi_j(s, z) z^k \, ds \, dz, \tag{6}$$

where $i+2r=l_0=\max(2m,m_j)$, K_j is the Poisson kernel of the B-elliptic problem constructed in (^{4}), so that for the finite function u we obtain the representation

$$D^i B_y^r u = D^i B_y^r v_N + D^i B_y^r w. \tag{7}$$

The uniqueness of the function u defined by (7), (6), can be proved according to the scheme of the proof of Theorem 4.1 of paper (^{2}).

2. Let l be an arbitrary integer $\geq l_0$. For any function v of the class C_B^l put

$$[v]_l = \sup |D^i B_y^r v|, \qquad |v|_l = \sum_{j=0}^{l} [v]_j,$$

where $i+2r=l$. The subclass of all functions from C_B^l for which the expressions $D^i B^r v$ uniformly satisfy the Hölder condition with exponent α $(0<\alpha<1)$ will be denoted by $C_B^{l+\alpha}$. For these functions we define the seminorm by the formula

$$[v]_{l+\alpha} = \sup \frac{|D^i B^r v(P) - D^i B^r v(Q)|}{|P-Q|^\alpha},$$

where the supremum is taken over all $i+2r=l$ and $P\neq Q$.

Theorem 1. Let $u(x,y,t)$ be a solution of problem (1)–(2), belonging to the class $C_B^{l_0+\alpha}$, and let it have compact support. If $f\in C_B^{l-2m+\alpha}$ in the quarter-space $y>0$, $t>0$, and $g_j\in C_B^{l-m_j+\alpha}$ on the half-plane $t=0$.

Then $u\in C_B^{l+\alpha}$,

$$[u]_{l+\alpha} \leq C\left([f]_{l-2m+\alpha} + \sum [g_j]_{l-m_j+\alpha}\right), \tag{8}$$

where C depends only on l,α and on the B-ellipticity constant of our problem.

Let $\Sigma^+ = \Sigma_R^+$ denote the quarter-ball:

$$|x|^2+y^2+t^2<R^2, \qquad y\geq 0, \quad t\geq 0$$

in (x,y,t)-space. Denote by σ_R^+ the flat part of the boundary: $t=0$, $y\geq 0$; d_P is the distance from the point P in Σ^+ to the spherical part of the boundary Σ^+, and $d_{P,Q}=\min(d_P,d_Q)$. By $\widehat{C}_B^l(\Sigma^+)$ denote the class of functions u for which the norm is finite

$$|\hat{u}|_{l+\alpha} = \sum_{j=0}^{l} \widehat{[u]}_j + \widehat{[u]}_{l+\alpha},$$

where

$$\widehat{[u]}_j = \sup d_P^j |D^i B_y^r u|,$$

$$\widehat{[u]}_{l+\alpha} = \sup d_{P,Q}^{l+\alpha} \frac{|D^i B^r u(P) - D^i B^r u(Q)|}{|P-Q|^\alpha}.$$

Theorem 2. Let u be a bounded solution of problem (1)–(2), belonging to the class $C_B^{l_0+\alpha}$. Suppose, moreover, that $f \in \widehat{C}_B^{l-2m+\alpha}(\Sigma^+)$ and $g_j \in \widehat{C}_B^{l-m_j+\alpha}(\sigma^+)$ for fixed $l \geq l_0$.

Then the function $u \in C_B^{l+\alpha}$ and the inequality holds

$$[\hat{u}]_{l+\alpha} \leq C\left(d_{P,Q}^{2m}[\hat{f}]_{l-2m+\alpha} + \sum d_{P,Q}^{m_j}[\hat{g}_j]_{l-m_j+\alpha} + [\hat{u}]_0\right), \tag{9}$$

where C does not depend on u, f, g_j, and R.

If growth of the solution at infinity with some rate is allowed, then Theorem 2 implies a generalization of Theorem 1.

Theorem 3. Suppose the quantity

$$M_0 = \lim_{R\to\infty} R^{-(l+\alpha)} \max_{\Sigma^+} |u|$$

is finite.

Then $u \in C_B^{l+\alpha}$ and the inequality holds

$$[u]_{l+\alpha} \leq C\left([f]_{l-2m+\alpha} + \sum [g_j]_{l-m_j+\alpha} + M_0\right). \tag{10}$$

3. The theorems given above make it possible, by means of the usual procedure of "freezing" the coefficients, to obtain analogues of Theorems 1 and 2 for equations with variable coefficients. In this case our equations have the form

$$\mathcal{L}(P, D, B_y)u(P) \equiv \sum_{i+2r\leq 2m} a_{i,r}(P) D^i B_y^r u(P) = F(P), \qquad t > 0, \tag{11}$$

$$H_j(P', D, B_y)u(P) = \sum_{i+2r\leq m_j} b_{i,r}^j(P') D^i B_y^r u(P) = G_j(P'), \qquad t = 0, \tag{12}$$

where $P = (x, y, t)$, $P' = (x, y, 0)$.

We also assume that the operators $\mathcal{L}, H_j$ and their coefficients satisfy certain conditions of type i)–iii) of work (2) (p. 70). We shall not formulate them, but note that they are essential.

Theorem 4. Let the function u belong to the class $C_B^{l+\alpha}$ and be a solution of the boundary-value problem (11)–(12) in the domain E_{n+2}^{++}.

Then

$$|u|_{l+\alpha} \leq C \left([F]_{l-2m+\alpha} + \sum [G_j]_{l-m_j+\alpha} + |u|_0 \right), \tag{13}$$

where C does not depend on u, F, and G_j.

Theorem 5. Let u be a solution of problem (11)–(12), belonging to the class $C_B^{l_0+\alpha}$ in Σ^+ for $R < 1$.

Then $u \in C_B^{l+\alpha}(\Sigma^+)$ and

$$|\hat{u}|_{l+\alpha} \leq C \left(d_{P,Q}^{2m} [\widehat{F}]_{l-2m+\alpha} + \sum d_{P,Q}^{m_j} [\widehat{G}_j]_{l-m_j+\alpha} + |\hat{u}|_0 \right), \tag{14}$$

where C does not depend on R.

4. In conclusion we give $L_{p,k}$-estimates for solutions of problems (1)–(2) and (11)–(12). Put, for $p > 1$ ($t = x_{n+2}$),

$$[u]_{j,L_{p,k}} = \left(\sum_{i=1}^{n} \int_{E_{n+2}^{++}} |D_{x_i}^j u|^p y^k \, dx \, dy + \int_{E_{n+2}^{++}} |D_t^j u|^p y^k \, dx \, dy + [B_y^{[j/2]} u]_{y,j-2[j/2],L_{p,k}}^p \right)^{1/p},$$

$$\|u\|_{l,L_{p,k}} = \left(\sum_{j=0}^{l} [u]_{j,L_{p,k}}^p \right)^{1/p}. \tag{15}$$

where

$$\left[B_y^{[j/2]} u\right]_{y,\, j-2[j/2],\, L_p,\, k} = \left(\int_{E_{n+2}^{++}} \left| B_y^{j/2} u \right|^p y^k \, dx \, dy \right)^{1/p}$$

in the case of even j. If j is odd ($j = 2r + 1$), then

$$\left[B_y^{[j/2]} u\right]_{y,\, 1,\, L_p,\, k} = \left(\int_{E_{n+2}^{++}} y^k \, dx \, dy \int_0^\infty \frac{\left| T_y^{2h} B^r u - 2T_y^h B^r u + B^r u \right|^p}{h^{1+p}} \, dh \right)^{1/p}.$$

For functions $\varphi(x, y)$ that are boundary values of functions $u(x, y, t)$, and for positive integers l, define the norms

$$\|\varphi\|_{l-1/p,\, L_p,\, k} = \inf \|u\|_{l,\, L_p,\, k},$$

where the infimum is taken over all such functions u that $u(x, y, 0) = \varphi(x, y)$.

Theorem 6. Let u be a solution of problem (1)–(2) and let it be finite. Suppose, moreover, that $\|u\|_{l,\, L_p,\, k} < \infty$, $l \geq l_0 + 1$.

Then

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$$\|u\|_{l,\,L_p,\,k} \le C\left(\|f\|_{l-2m,\,L_p,\,k} + \sum \|g_j\|_{l-m_j-1/p,\,L_p,\,k}\right), \tag{16}$$

where the constant C depends only on l, p, and the constant of B-ellipticity of our problem.

Theorem 7. Let u be a solution of problem (11)–(12) and tend to zero as $|P| > \rho$, where ρ is sufficiently small. Suppose that the norm $\|u\|_{l_0+1,\,L_p,\,k}$ is finite.

Then the norm $\|u\|_{l,\,L_p,\,k}$ for $l \ge l_0 + 1$ is also finite and

$$\|u\|_{l,\,L_p,\,k} \le C\left(\|F\|_{l-2m,\,L_p,\,k} + \sum \|G_j\|_{l-m_j-1/p,\,L_p,\,k} + \|u\|_{0,\,L_p,\,k}\right), \tag{17}$$

where C does not depend on u, F, G_j.

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