ON THE PROBLEM OF THE GRAVITATIONAL STABILITY OF A DUST CLOUD

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Abstract

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ON THE PROBLEM OF THE GRAVITATIONAL STABILITY OF A DUST CLOUD

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0. Statement of the problem. We consider the equation

$$\partial f/\partial t + v^k \partial f/\partial x^k + F^k(x)\partial f/\partial v^k = 0; \quad f = f(x, v, t); \quad x = (x^1, x^2, x^3);$$

$$v = (v^1, v^2, v^3); \quad F^k(x) = -\frac{\partial \Phi}{\partial x^k}; \quad \Phi(x) = -\gamma \iint \frac{f(y, v)}{|x-y|}\, dy\, dv. \tag{1}$$

The stability of a stationary solution of equation (1) is studied. We shall restrict ourselves to considering the simplest stationary solutions of the form

$$f^0(x, v) = h^0(\varepsilon^0(|x|, |v|)); \quad \varepsilon^0(|x|, |v|) = |v|^2/2 + \Phi^0(|x|). \tag{2}$$

§ 1. **Linear consideration.**

1. Linearization.

Putting $f = f^0 + \varepsilon g$, we obtain

$$\partial g/\partial t + \hat{A} g - \hat{B} g = 0, \tag{3a}$$

where

$$\hat{A} g = v^k \frac{\partial g}{\partial x^k} + F^{0k}(x)\frac{\partial g}{\partial v^k}; \quad \hat{B} g = \gamma \frac{\partial f^0}{\partial v^k} \frac{\partial}{\partial x^k} \left(\iint \frac{g(y, v)}{|x-y|}\, dy\, dv \right). \tag{3b}$$

Following V. A. Antonov (1), we put $g = g_+ + g_-$; $g_+ = \frac{1}{2}(g(x, v, t) + g(x - v, t))$. Since $\hat{B} g_- = 0$, the consequence of (3) is the following equation for $\psi = g_-$

$$\partial^2 \psi/\partial t^2 = \hat{K} \psi, \quad \text{where} \quad \hat{K} = -\hat{A}^2 + \hat{B}\hat{A}. \tag{4}$$

2. The condition $h^{0\prime}(\varepsilon) < 0$ and the quadratic form K. Let $h'(\varepsilon^0) < 0$ in the domain Ω, where $h(\varepsilon^0) = f^0(x, v) \neq 0$. The operator $\hat{K}$ is self-adjoint (see ([1])) in the scalar product

$$(\psi_1, \psi_2) = \int_\Omega \frac{\psi_1(x, v)\psi_2(x, v)}{|h'(\varepsilon^0(x, v))|}\, dx\, dv.$$

(The functions ψ are considered only on Ω.) For stability of equation (4) when $h'(\varepsilon) < 0$, it is necessary and sufficient that

$$K(\psi, \psi) = (\hat{K}\psi, \psi) \geq 0. \tag{5}$$

More precisely: condition (5) guarantees the absence of solutions of (4) and (3) growing faster than t. In equation (3) there are obvious growing solutions corresponding to uniform translational motion of the cloud (see ([1])). Conditions (5) are insufficient to guarantee the absence of other growing solutions of equation (3). Below, when speaking of stability, we have in mind the nonnegativity of $K(\psi, \psi)$. The difficult question of linearly growing solutions of (3) is left open here.

§ 3. On the simplest sufficient condition of nonnegativity

$$K(\psi, \psi) = \Gamma(\beta, \beta) =$$

$$= -\int \frac{\beta^2(x, v)}{h'(\varepsilon^0(x, v))}\, dx\, dv - \gamma \int \frac{\beta(x_1, v_1)\beta(x_2, v_2)}{|x_1 - x_2|}\, dx_1\, dv_1\, dx_2\, dv_2, \tag{6}$$

where

$$\beta = A\psi = v^k \partial\psi/\partial x^k + F^{0k}(x)\partial\psi/\partial v^k; \qquad \psi(x, v) = -\psi(x, -v). \tag{7}$$

Sufficient conditions for the nonnegativity of $K(\psi, \psi)$ are obtained from the requirement $\Gamma(\beta, \beta) \geq 0$ on a class of functions β broader than (7). The crudest sufficient condition is obtained if one requires $\Gamma(\beta, \beta) \geq 0$ for all β. Fix $\int \beta(x, v)\, dv = \alpha(x)$ and solve the variational problem for the minimum of $\Gamma(\beta, \beta)$ under this additional condition. We obtain

$$\min \Gamma(\beta, \beta) = G(\alpha, \alpha) = \int \frac{\alpha^2(x)}{a^0(x)}\, dx - \gamma \int \frac{\alpha(x)\alpha(y)}{|x - y|}\, dx\, dy =$$

$$= \int \left[\frac{\alpha(x)}{a^0(x)} + 4\pi\gamma\Delta^{-1}\alpha(x)\right] \alpha(x)\, dx^*, \tag{8a}$$

where

$$a^0(x) = -\int h'(\varepsilon^0(x,v))\,dv = \frac{\rho^{0\prime}(r)}{\Phi^{0\prime}(r)}; \qquad \rho^0(r) = \int f_-^0(x,v)\,dv; \qquad |x| = r. \tag{8 }$$

The condition $G(\alpha,\alpha) \geq 0$ for all α is the simplest sufficient condition for the nonnegativity of $K(\psi,\psi)$. It can be rewritten as the condition of nonnegativity of the operator $1/a^0 + 4\pi\gamma\Delta^{-1} \geq 0$, or in the equivalent form

$$\hat{P}_0 = -\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - 4\pi\gamma a^0(r) \geq 0^{**} \tag{9}$$

(see (2)). Let us now note that the equation $\hat{P}_1\alpha = 0$, where $\hat{P}_1 = \hat{P}_0 + 2/r^2$, has the solution

$$\alpha^0(r) = \frac{1}{r^2}\int_0^r \rho^0(r)r^2\,dr.$$

Since

$$\int_0^\infty (\alpha^0(r))^2 r^2\,dr < \infty,$$

$\lambda = 0$ is an eigenvalue of $\hat{P}_1$. Since $\hat{P}_0 < \hat{P}_1$, condition (9) is never satisfied. It is not difficult to indicate other sufficient conditions; however, their effectiveness needs to be checked.

Remark. Since $a^0(r) \neq 0$ for $r > 0$, by virtue of the known oscillation theorems (see, for example, (^{3})), $\hat{P}_1 \geq 0$.

§ 4. Radial perturbations

The function $\overline{\Phi}(x,v)$, invariant under a simultaneous rotation in the spaces x and v, is represented in the form

$$\overline{\psi}(x,v) = \overline{\psi}(|x|,|v|,(x,v)); \qquad (x,v) = \sum_1^3 x^k v^k. \tag{10}$$

For brevity we shall agree to call such functions radial. An arbitrary function

$$\psi(x,v) = \overline{\psi}(x,v) + \tilde{\psi}(x,v), \tag{11}$$

where $\overline{\psi}(x,v)$ is radial, and $\int \tilde{\psi}(x,v)\mu(x,v)\,dx\,dv = 0$ for any radial μ. If one restricts oneself to considering radial perturbations, then one may require $K(\psi,\psi) > 0$***.

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* Δ^{-1} is the operator inverse to the Laplace operator in all space.

** If P and Q are two self-adjoint operators and $Q > 0$, then from $P > Q$ it follows that $P^{-1} < Q^{-1}$. The proof follows easily from writing $P - Q = P^{1/2}(1-C)P^{1/2}$, $P^{-1} - Q^{-1} = P^{-1/2}(1-C^{-1})P^{-1/2}$, where $C = P^{-1/2}QP^{-1/2}$.

*** Considering $\psi = -\psi(|x|,|v|)$ does not make sense, since such functions are not invariant with respect to (3).

p. 5. Lemma. Let $\alpha(x) = \int \beta(x,v)\,dv$, and suppose that in the expansion of $\alpha(x)$ in spherical harmonics there is no zeroth term:

$$\alpha(x) = \sum_{l=1}^{\infty} \alpha_{lm}(r) Y_{lm}(\theta,\varphi). \tag{12}$$

$$|m| < l$$

Then $\Gamma(\beta,\beta) > 0$.

Proof. The minimum of $\Gamma(\beta,\beta)$ under the condition $\int \beta(x,v)\,dv = \alpha(x)$ is given by formula (8). Substituting (12) into (8), we obtain

$$\min \Gamma(\beta,\beta) = G(\alpha,\alpha) = \sum_{l=1}^{\infty} G_l(\alpha_{lm},\alpha_{lm}), \tag{13a}$$

$$|m| < l$$

where

$$G_l(\alpha,\alpha) = \int \left(\frac{\alpha(r)}{a^0(r)} + 4\pi\gamma\Delta_l^{-1}\alpha \right) \alpha(r)\,dr,$$

$$\Delta_l = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{l(l+1)}{r^2}. \tag{13b}$$

For $l > 1$, $\Delta_l^{-1} > \Delta_1^{-1}$, and hence $G_l(\alpha,\alpha) > G_1(\alpha,\alpha)$. We now note that the operator $1/a^0(r) + 4\pi\gamma\Delta_1^{-1} \geq 0$, since $-\Delta_1 - 4\pi\gamma a^0(r) \geq 0$ (see the remark in p. 4). Consequently, $G_1(\alpha,\alpha) \geq 0$, and therefore $G(\alpha,\alpha) \geq 0$.

p. 6. Reduction to radial perturbations.

Theorem. If the solution $f^0(x,v) = h(\varepsilon^0(x,v))$, with $h'(\varepsilon) < 0$, is stable with respect to radial perturbations, then it is stable with respect to arbitrary perturbations*. In other words: if $K(\bar{\psi},\bar{\psi}) \geq 0$ for any radial $\bar{\psi}$, then $K(\psi,\psi) \geq 0$ for any ψ.

Proof. Write the expansion $\psi = \bar{\psi} + \tilde{\psi}$ (see (11)). It is easy to verify that $K(\psi,\psi) = K(\bar{\psi},\bar{\psi}) + K(\tilde{\psi},\tilde{\psi})$. Every function $\tilde{\psi}$ satisfies the conditions of the lemma, since $\int \tilde{\psi}\,\mu(r)\,dx\,dv = 0$ for any $\mu(r)$. Consequently, $K(\tilde{\psi},\tilde{\psi}) \geq 0$. The theorem is proved.

§ 2. Nonlinear consideration. A nonlinear treatment of the problem seems desirable, since in neutral problems (those without a distinguished direction of time) a linear treatment cannot give a final answer to the question of stability**.

p. 7. Variational principle. Consider Hamiltonian variations of the original function

$$f^0 \to f_\tau = f^0(x(\tau), v(\tau)); \qquad dx/d\tau = \partial H/\partial v; \qquad dv/d\tau = -\partial H/\partial x; \quad (14)$$

$H(x,v,\tau)$ is an arbitrary function of 7 variables. Let $E[f]$ be the total energy of the system,

$$E[f] = \int \frac{|v|^2}{2} f(x,v)\,dx\,dv - \frac{\gamma}{2}\int \frac{f(1)f(2)}{|x_1 - x_2|}\,d1\,d2. \quad (15)$$

We shall show that $E[f]$ has a conditional extremum at f^0, and find the conditions under which this extremum is a minimum. For small τ,

$$f_\tau = f^0 + \tau\delta f + \tfrac{1}{2}\tau^2\delta^2 f + o(\tau^2),$$

$$\delta f = \hat{H}_0[f^0]; \qquad \delta^2 f = \hat{H}_0^2[f^0] - \hat{H}_1[f^0]; \qquad H_0(x,v,\tau) = H(x,v,0); \quad (16)$$

$$H_1(x,v,0) = \frac{\partial}{\partial\tau}H(x,v,0); \qquad \hat{H}[f] = \frac{\partial H}{\partial v}\frac{\partial f}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial f}{\partial v}.$$

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* This assertion is analogous to the following theorem, proved recently by one of the authors: for the stability of a gas sphere in its own gravitational field it is necessary and sufficient that: 1) it be stable with respect to radial perturbations; 2) the condition of absence of convection hold.

** In addition, in the linear treatment of this problem we were forced to restrict ourselves to perturbations $g = \delta f$ concentrated where $f^0(x,v) \neq 0$. This restriction is in no way required by the substance of the problem.

Substituting these expressions into (15), we obtain

$$E_\tau = E^0 + \tau\delta E + {}^1\!/_2\tau^2\delta^2 E + o(\tau^2),$$

where

$$\begin{aligned} \delta E &= \int \left(\frac{|v|^2}{2} + \Phi^0(|x|)\right)\delta f\,dx\,dv = \int \varepsilon^0\delta f\,dx\,dv, \\ \delta^2 E &= \int \varepsilon^0\delta^2 f\,dx\,dv - \gamma\int \frac{\delta f(1)\,\delta f(2)}{|x_1 - x_2|}\,d1\,d2. \end{aligned} \tag{17}$$

Using: a) the antisymmetry of the operators $\hat{H}_0,\ \hat{H}_1;\ \hat{A}$; b) $\hat{H}[h(\varepsilon)] = h'(\varepsilon)\hat{H}[\varepsilon]$; c) $\hat{A}[h(\varepsilon^0)] = \hat{A}[\varepsilon^0] = 0$, we obtain

$$\delta E = \int \varepsilon^0 h'(\varepsilon^0)\hat{H}[\varepsilon^0]\,dx\,dv = -\int \varepsilon^0 h'(\varepsilon^0)\hat{A}[H]\,dx\,dv = 0;$$

$$\delta^2 E = -\int h'(\varepsilon^0)b^2(x,v)\,dx\,dv - \gamma\int \frac{b(1)b(2)}{|x_1 - x_2|}h'(\varepsilon^0(1))h'(\varepsilon^0(2))\,d1\,d2;$$

$$b(x,v) = \hat{H}_0[\varepsilon^0(x,v)].$$

It is easy to see that for nonnegativity of $\delta^2 E$ it is necessary that $h'(\varepsilon^0) \leq 0$. If $h'(\varepsilon^0) < 0$, then $\delta^2 E = \Gamma(\beta,\beta)$, where $\beta = \hat{H}[f^0] = -\hat{A}[h'(\varepsilon^0)H_0] = \hat{A}\psi$, and $\Gamma(\beta,\beta)$ is the quadratic form already known to us, (6). Thus: 1) for the Hamiltonian variations (14), $\delta E = 0$; 2) the conditions $\delta^2 E \geq 0$ for these variations coincide with the conditions of linear stability $h'(\varepsilon) < 0;\ K \geq 0$*.

§ 8. **Variational principle and stability.** The set Q of all f_τ (for all possible $\hat{H}$)** is invariant with respect to equation (1). For ordinary differential equations, from the fact that a certain integral of the equations E has a minimum on the invariant surface Q at the (stationary) point f^0, the stability of f^0 automatically follows (cf. with (4)). A complete proof of the stability of f^0 in our case might consist (as in the finite-dimensional case) in constructing a Lyapunov function, i.e. such a functional $\Lambda[f]$ that $d\Lambda/dt = 0$ by virtue of (1) and $\Lambda[f^0] = \min$. In view of the neutrality of our problem, it must be $d\Lambda/dt = 0$, i.e. Λ is an integral of equation (1). Besides the energy $E[f]$ (15), we know only the following integrals:

$$I[f] = \int J(f(x,v))\,dx\,dv$$

($J(f)$ is an arbitrary function). We have not succeeded in constructing $\Lambda[f]$ from these integrals. It seems to us that this situation is typical for neutral infinite-dimensional problems: information about the first integrals is often insufficient for constructing a Lyapunov function, and one has to restrict oneself to a variational principle in the spirit of § 7 and finite-dimensional analogies. From this point of view, the problems considered in (4) (and the problem of a gravitating gas sphere), where such a construction is possible, are rather the exception than the rule.

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* The variational principle formulated is also valid for stationary solutions of a more general type.

** More precisely: Q consists of all $f(x,v) = f^0(X(x,v), V(x,v))$, where $x, v \to X, V$ is an arbitrary canonical transformation.

Note: Figure translations are in progress. See original paper for figures.