ON FINITE SEPARABILITY IN SEMIGROUPS

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Abstract

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ON FINITE SEPARABILITY IN SEMIGROUPS

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A semigroup S is called a **semigroup with finitely separable subsets** if, for any subset M of S and any $a \in S \setminus M$, there is a homomorphism φ of the semigroup S into a finite semigroup such that $\varphi(a) \notin \varphi(M)$.

If, instead of all subsets in this definition, one considers only subsemigroups of S, then one arrives at the definition of a **finitely separable semigroup** ([1]).

This note gives a necessary and sufficient criterion describing semigroups with finitely separable subsets (Theorem 1). Necessary and sufficient conditions are also given for finite separability of semigroups from classes often encountered in the literature. Semigroups with finitely separable subsets are, obviously, finitely separable. In some cases the converse is also true: finitely separable semigroups turn out to be semigroups with finitely separable subsets (this, for example, holds for the semigroups considered in Theorems 3, 4, 7). For such semigroups, the necessary and sufficient conditions for finite separability are therefore contained in Theorem 1.

Among the corollaries of the theorems given in the note are, in essence, all results on finite separability of semigroups known up to now ([5–8]).

Let S be a semigroup and let $a, b \in S$. Put

$$[a : b] = \{(u, v) \in S^1 \times S^1 \mid ubv = a\},$$

where S^1 is the semigroup obtained from the semigroup S by adjoining an identity externally.

Theorem 1. In order that a semigroup S be a semigroup with finitely separable subsets, it is necessary and sufficient that, for each fixed a from S, among the sets $[a : x]$, where x ranges over S, only a finite number be distinct.

Theorem 2. In order that a commutative semigroup S be finitely separable, it is necessary and sufficient that the following two conditions be satisfied:

(a) every maximal subgroup of S is periodic, and the orders of the elements of each of its primary components are bounded in the aggregate;

(b) for any nonregular element $a \in S$, among the sets $[a : x]$, where x ranges over S, only a finite number are distinct.

Corollary 1. In order that a commutative goloid ([2]) semigroup be finitely separable, it is necessary and sufficient that it be an ordinal sum ([3]) of monogenic semigroups.

Corollary 2 ([5]). In order that a commutative regular semigroup S be finitely separable, it is necessary and sufficient that every maximal subgroup of S be periodic and that the orders of the elements of each of its primary components be bounded in the aggregate.

Corollary 3. A commutative Archimedean ([4]) semigroup with a finite number of generators is finitely separable if and only if it either has no idempotents or is finite.

Denote by Π the class of semigroups S such that, for each $u, a, v \in S$, from $a = uav$ it follows that $u \notin aS^1,\ v \notin S^1a$.

Theorem 3. In order that a semigroup S belonging to the class Π be finitely separable, it is necessary and sufficient that S be a semigroup with finitely separable subsets.

Corollary 1. In order that a semigroup without idempotents be finitely separable, it is necessary and sufficient that it be a semigroup with finitely separable subsets.

Corollary 2*. In order that a nilsemigroup S be finitely separable, it is necessary and sufficient that, for each of its nonzero elements a, among the sets $[a : x]$, where x ranges over S, there be only a finite number of distinct ones.

Corollary 3. In order that a nilsemigroup in which the weak cancellation law holds be finitely separable, it is necessary and sufficient that each of its nonzero elements have only a finite number of distinct divisors.

Theorem 4. A semigroup S in which the left cancellation law holds is finitely separable only in the following cases:

(a) if S is nonregular, then it is a semigroup with finitely separable subsets;

(b) if S is regular, then it is the direct product of a periodic finitely separable group and a semigroup of right zeros.

Corollary. A semigroup S with the two-sided cancellation law is finitely separable only in the following cases:

(a) if S is nonregular, then each of its elements has only a finite number of distinct divisors;

(b) if S is regular, then it is a periodic finitely separable group.

Theorem 5. Let S be a commutative semilattice of semigroups, in each of which the two-sided cancellation law holds. In order that S be finitely separable, it is necessary and sufficient that the following two conditions be satisfied:

(a) the maximal subgroups of S are periodic and finitely separable;

(b) for every nonregular element $a \in S$, among the sets $[a : x]$, where x ranges over S, there are only finitely many distinct ones.

Corollary (*). In order that a completely regular inverse semigroup be finitely separable, it is necessary and sufficient that all its maximal subgroups be periodic finitely separable groups.

Let S be a completely simple semigroup without zero (with zero). As is known ([2],[4]), S is isomorphic to the so-called regular matrix semigroup over some group G (over a group G with an externally adjoined zero), with defining matrix $P = \|p_{\lambda i}\|$ (λ ranges over some index set Λ, i over the index set I). Such a semigroup, following ([4]), will be denoted by $M(G; I, \Lambda; P)$ (respectively by $M^0(G; I, \Lambda; P)$).

Let N be a normal divisor of the group G. The matrix obtained from P by replacing all elements by the corresponding cosets modulo N will be called the **factor matrix** and denoted by P/N.

Two rows of the matrix P with "numbers" λ and μ, respectively ($\lambda, \mu \in \Lambda$), will be called **proportional on the left (on the right)** if there exists an element $g \in G$ such that

$$p_{\lambda i} = g p_{\mu i} \qquad (p_{\lambda i} = p_{\mu i} g)$$

for all $i \in I$.

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* The assertions contained in Corollaries 2 and 3 of Theorem 3 and in the corollary to Theorem 4 were also obtained independently by S. T. Mamikonian ([6]).

Proportionality on the left and on the right for the columns of the matrix P is defined analogously.

Obviously, the relation of proportionality on the left is an equivalence relation on the set of all rows of the matrix P. We shall denote by m_1 the cardinality of the corresponding quotient set. By m_2 we shall denote the analogous cardinality for the relation of proportionality on the right on the set of columns of the matrix P. If both cardinalities m_1, m_2 are finite, then we shall say that the matrix P has **finite rank**.

Theorem 6. In order that the semigroup $M(G; I, \Lambda; P)$ or $M^0(G; I, \Lambda; P)$ be a semigroup with finitely separable subsets, it is necessary and sufficient that the group G be finite and the matrix P have finite rank.

Theorem 7. In order that the semigroup $M^0(G; I, \Lambda; P)$, having nontrivial zero divisors, be finitely separable, it is necessary and sufficient that it be a semigroup with finitely separable subsets.

Theorem 8. In order that the semigroup $M(G; I, \Lambda; P)$ be finitely separable, it is necessary and sufficient that G be a periodic finitely separable group and that, for every normal divisor N of finite index, the quotient matrix P/N have finite rank.

Theorem 9. Let the semigroup S be the free product of the semigroups S_i ($i \in I,\ |I| \geq 2$). Then the following conditions are equivalent:

(a) S is finitely separable;

(b) all the semigroups S_i are semigroups with finitely separable subsets;

(c) S is a semigroup with finitely separable subsets.

Some of the results of this note were reported by the author at the IX All-Union Algebraic Colloquium ([7]).

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CITED LITERATURE

1. A. I. Mal' tsev, *Uch. zap. Ivanovsk. ped. inst.*, **18**, 49 (1958).
2. E. S. Lyapin, *Semigroups*, M., 1960.
3. L. N. Shevrin, *Izv. vyssh. uchebn. zaved., Matematika*, **6**, 156 (1965).
4. A. H. Clifford, G. B. Preston, *The Algebraic Theory of Semigroups*, Am. Math. Soc., Providence, 1, 1961; 2, 1967.
5. M. M. Lesokhin, *Matem. sborn.*, **74** (116), 2, 314 (1967).
6. S. T. Mamikonyan, XXI Herzen Readings, Mathematics, 1968, p. 27.
7. E. A. Golubov, IX All-Union Algebraic Colloquium, Abstracts of Scientific Communications, Gomel, 1968, p. 59.
8. M. M. Lesokhin, *Uch. zap. Leningradsk. ped. inst.*, **387**, 134 (1968).

Note: Figure translations are in progress. See original paper for figures.