NONLINEAR RELATIVISTIC WAVES IN A SUPERCOMPRESSED GAS

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Abstract

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NONLINEAR RELATIVISTIC WAVES IN A SUPERCOMPRESSED GAS

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Let us pose the Cauchy problem for an ideal gas with equation of state $\varepsilon = 3p$, with the following initial distributions of velocity $v_0(x)$ and pressure $p_0(x)$: $v_0(x) = \beta\nabla r$ ($|\beta| < c$), $p_0(x) = \alpha/r^{4l}$, where c is the speed of light and α is a characteristic dimensional constant. The argument r will be understood as the distance, respectively, from the center and from the axis of symmetry. Under such initial conditions the subsequent motion will be self-similar and will possess, respectively, point and axial symmetry. We introduce the dimensionless parameter $\lambda = r/ct$ and denote $v(r,t) = cV(\lambda)$, $p(r,t) = p_0(r)P(\lambda)$. The system of equations of conservation of momentum and energy $\nabla_i T^{ki} = 0$ for an ultrarelativistic gas, for spherical and cylindrical waves, is then reduced to the form

$$dV/d\lambda = (1-V^2)\,[3l\lambda + (k-3l)V^2\lambda - kV]/\lambda(1+\lambda^2V^2+4V\lambda-3\lambda^2-3V^2), \quad (1)$$

where the value of the parameter k in the spherical case is equal to 2, and in the cylindrical case to 1. Given the known function $V(\lambda)$, the dimensionless pressure $P(\lambda)$ can be found by quadrature of the equation

$$d\ln P = 4[\lambda(\lambda - V)dV + l(1-V^2)d\lambda]/\lambda(1-V^2)(1-\lambda V). \quad (2)$$

The conditions connecting the dimensionless velocity of the gas in front of and behind the shock wave will be written out from paper ([2])*

$$V_2 = (3\lambda^2 - 1 - 2V_1\lambda)/[V_1(\lambda^2-3) + 2\lambda]. \quad (3)$$

This transformation takes the straight line $V = 1$ into the curve NB: $(3\lambda + 1)/(3+\lambda) = V$ (Fig. 1). The image of the straight line $V = -1$ under transformation (3) is the curve MB: $(3\lambda - 1)/(3-\lambda) = V$. On the curves EB: $(\lambda\sqrt{3}+1)/(\sqrt{3}+\lambda) = V$ and FB: $(\lambda\sqrt{3}-1)/(\sqrt{3}-\lambda) = V$ lie points that,

Fig. 1

Figure 1: Fig. 1

after the discontinuity, transform into themselves. The points $(0, 1)$, $(0, -1)$, $O(0, 0)$ in the (λ, V) plane are singular (of saddle type). The separatrix of the saddle O has the asymptote $V = \dfrac{3l}{k+1}\lambda$. The characteristic pattern of integral curves depends on the type of symmetry of the problem and on the value of the parameter l.

Fig. 1

Let us list the properties of the integral curves of equation (1), qualitatively identical for the spherical and cylindrical cases. If

* In this paper spherical waves were studied for the case $l = 0$, but an error made in the sign before the right-

$0 < l < (1 - \sqrt{3}/3)k/2$ (Fig. 1), then there exists a singular point A (a singularity of node type) with coordinates $(k\sqrt{3}/3(k-2l),\ \sqrt{3}l/(k-3l))$. All integral curves enter the point A, touching the separatrix of the node A with asymptotic form

$$\left(V - \frac{\sqrt{3}l}{k-3l}\right) = \left(\lambda - \frac{k\sqrt{3}}{3(k-2l)}\right)\frac{(2l-k)^2}{8k(3l-k)^2}\left(-a + \sqrt{a^2 - 144kl(3l-k)}\right),$$

where $a = 3k^2 - 12lk - 6k$. The asymptotic form of the other separatrix of the point A differs from the expression given only by the sign in front of the root. As l increases, the singular point A moves along the curve $V = (\lambda\sqrt{3} - 1)/(\sqrt{3} - \lambda)$ from the point $(\sqrt{3}/3, 0)$ (at $l = 0$) to the point $B\ (1, 1)$ (when $l = (1 - \sqrt{3}/3)k/2$). In the interval $(1 - \sqrt{3}/3)k/2 \le l \le (1 + \sqrt{3}/3)k/2$ (Fig. 2) the singular point A is absent. When $l > (1 + \sqrt{3}/3)k/2$, but $a^2 > 144kl(3l - k)$, the singular point A $(k\sqrt{3}/3(2l-k),\ \sqrt{3}l/(3l-k))$ appears again. As l increases, the node A moves from the point $B\ (1, 1)$ along the curve $V = (\lambda\sqrt{3} + 1)/(\sqrt{3} + \lambda)$ to a certain critical point $\tilde{A}$ at $l = \tilde{l}$, where $\tilde{l}$ is the root of the equation $a^2 = 144k\tilde{l}(3\tilde{l} - k)$. If l varies in the interval $((1 + \sqrt{3}/3)k/2, \tilde{l})$, then the curves enter the point A, touching the separatrix of the node A with asymptotic form

$$\left(V - \frac{\sqrt{3}l}{3l-k}\right) = \left(\lambda + \frac{k\sqrt{3}}{3(k-2l)}\right)\frac{(2l-k)^2}{8k(3l-k)^2}\left(-a - \sqrt{a^2 - 144kl(3l-k)}\right).$$

For the other separatrix of the node A, the asymptotic form differs only by the sign in front of the root. For all $l \in (0, \tilde{l})$, the separatrix of the saddle $O\ (0, 0)$

abuts B with asymptotic form $(\lambda-1)\{2+k/2-3l+\sqrt{(2+k/2-3l)^2+k-1}\} = V-1$. In the intervals $0 < l < (1-\sqrt{3}/3)k/2$ and $(1+\sqrt{3}/3)k/2 < l < \tilde{l}$, the separatrix of the saddle O, passing through the node A, coincides with that separatrix of the point A which the integral curves touch. For $l \in ((1-\sqrt{3}/3)k/2, (1+\sqrt{3}/3)k/2)$, part of the integral curves enter the point B, touching the separatrix OB. The node A at $l = \tilde{l}$ degenerates (not dicritically). For $l > \tilde{l}$ the singular point A becomes a focus, moving with increasing l along the curve $V = (\lambda\sqrt{3}+1)/(\sqrt{3}+\lambda)$ from $\tilde{A}$ to $(0, \sqrt{3}/3)$ as $l \to \infty$. In the spherical case ($k = 2$), almost all integral curves at the point B, for all l, touch the straight line $V = 1$ with asymptotic form $V - 1 = \mathrm{const}(\lambda-1)^2$. The asymptotic form of the separatrices at the point B is as follows: $V - 1 = (\lambda-1)[3(1-l) \pm \sqrt{3(1-l)^2+1}]$. In the cylindrical case ($k = 1$), the integral curves (with the exception of the separatrix and the straight line $V = 1$) enter the point B, touching the straight line $V = 1$ only from one side (asymptotic form $V - 1 = (\lambda-1)(6l-5)^{-1}(\mathrm{const} + \ln|\lambda-1|)^{-1}$: for $l < 5/6$, only from the left; for $l > 5/6$, only from the right. At $l = 5/6$ the integral curves do not enter the point B. For all l, the point B has only one separatrix; its asymptotic form at B is as follows: $(V-1) = (5-6l)(\lambda-1)$. The character of the point A at $l = 0$ is different in the spherical and cylindrical cases: for $k = 2$ the point A (Fig. 3) is a degenerate node (with asymptotic form of the integral curves $V = (3\lambda-\sqrt{3}/(\ln|\lambda-\sqrt{3}/3| + \mathrm{const})))$; for $k = 1$ it is an ordinary node. All integral curves for $k = 1$ enter the point A, touching the straight line $V = (3\lambda-\sqrt{3})/4$. The qualitative study of equation (1) just presented admits a twofold physical interpretation.

I. **Ejection of matter from a singular point (axis) at the instant** $t = 0$.* At a fixed instant t, the matter ejected from the point can be located only within a sphere (cylinder) of radius ct and center at the center (axis) of symmetry. The difference in the modes of ejection of matter is regulated by changing the parameter l. The separatrix OB gives the velocity distribution at a fixed instant for $l \in (0, \tilde{l})$. If $l > \tilde{l}$

* The author owes the idea to I. S. Shkian.

($\bar{l} = 2$ for $k = 2$ and $\bar{l} = (3+\sqrt{11})/8$, $l < 5/6$ for $k = 1$), the motion occurs with a shock wave: the representative point jumps from one separatrix to another and follows it to the point $B(1, 1)$. For the pressure, for all l near the center (axis) ($\lambda \ll 1$) of symmetry, we have the expansion $p(r,t) = \frac{\alpha}{t^{4l}}\left[1 + \beta\frac{r}{t} + \ldots\right]$, where $\beta = \frac{12l}{c(k+1)^2}(lk - 2l + k + 1)$. For $l < 5/6$ for axial symmetry and for all l for point symmetry, near $B(1, 1)$ the pressure has the asymptotic form: if $k = 2$,

$$p = \frac{\alpha}{r^{4l}} \exp\left[\ln\left|\lambda - 1 - \frac{2\left(l-2-\sqrt{9(l-1)^2+1}\right)}{-3l+4+\sqrt{9(l-1)^2+1}}\right|\right],$$

Figure 2 and Figure 3

Figure 2: Figure 2 and Figure 3

if $k = 1$,

$$p = \frac{\text{const}}{r^{4l}}(-\lambda + 1)^{-1/3}.$$

From these expressions it is clear that the matter is concentrated mainly near the sphere (cylinder) expanding with the speed of light. For $l >$

Fig. 2

Fig. 3

5/6 in the cylindrical case, ejection of matter from the axis is likewise impossible without a shock wave; however, the scheme of motion becomes different: behind the shock wave all the matter moves with the speed of light. Outside the core bounded by the shock wave, in which the velocity distribution is represented by a piece of the separatrix of the saddle from $O(0, 0)$ to the point of intersection with the image of $V = 1$—the curve $V = (3\lambda + 1)/(3 + \lambda)$, the pressure distribution is given by the formula:

$$p = \frac{\alpha}{r^{4l}}\left(\frac{r}{ct - r}\right)^{2-4l}.$$

II. **Solution of the Cauchy problem.** For $l \in (0, (k + 1)/3)$ there exists a solution of the problem of focusing at a point. A spherical (cylindrical) core is formed, bounded by a shock wave and expanding with constant velocity (for $l = 0$, less than $c\sqrt{3}/3$). Inside the core the velocity distribution is given by a piece of the separatrix of the saddle $(0, 0)$ from the point $(0, 0)$ to the point of intersection with the curve $V = (3\lambda - 1)/(3 - \lambda)$ (the image of $V = -1$). For $l = 0$ the matter inside the core is at rest ([1])*. For $l \geq (k+$

* References to the monograph ([1]) are intended to emphasize the qualitative analogy between the conclusions of this paper and the corresponding first principal results of L. I. Sedov in nonrelativistic gas dynamics.

$+1)/4$, there is no physical solution of the problem of focusing with a subcritical velocity.

The expansion problem. When $k = 2$, there exists for all l a certain critical expansion velocity V^*, which for $l < 5/6$ does not exist when $k = 1$. The motion of the expansion (with subcritical initial velocities for $k = 2$ and arbitrary initial velocities > 0 for $k = 1$) for $l \in (0, (\sqrt{3}/3 + 1)k/2)$ occurs with a weak discontinuity ([1]). When followed from infinity to the center, the velocity first increases to a certain value $\hat{V} = \max V(\lambda)$ (lying on the curve $3l\lambda + (k -$

$3l)\hat{V}^2\lambda - k\hat{V} = 0)$, and then falls to $V = \sqrt{3}l/(k-3l)$ at $r = k\sqrt{3}ct/3(k-2l)$. At this value of r a weak discontinuity occurs. Inside the spherical core ($r < \sqrt{3}kct/3(k-2l)$), for fixed t, the motion is standard for all initial velocities: it is given by a segment of the separatrix of the saddle $(0,0)$ up to the point $A(k\sqrt{3}/3(k-2l), \sqrt{3}l/k-3l)$. For $l \in ((1-\sqrt{3}/3)k/2, (k+1)/4)$, during the expansion of the gas (with subcritical initial velocities for $k=2$ and arbitrary initial velocities for $k=1$) a shock wave is formed: the representative point on the separatrix of the saddle $(0,0)$ arrives by a jump. In all solutions for $l \geq (k+1)/3$, the pressure tends to ∞ on approaching the sphere (cylinder) $r = ct$, both from outside and from inside. We consider this case separately for point and axial symmetry.

1) $k=2$. In the case when $l=1$, the separatrix OB has the equation $V = \lambda$, and the other separatrix of the point B has the equation $V = 1/\lambda$. Thus, analysis of the solution of the Cauchy problem with an initial pressure distribution according to the law $p_0 = a/r^4$ for a gas at rest leads to the conclusion that inside the light sphere the solution is given by the well-known cosmological solution of Milne ([3]), while outside it by the formulas $v = c/\lambda = c^2t/r$, $p = a/(r^2-t^2)^{2/3}r^{8/3}$. If $l \in (1, 5/4]$, then the solution of the Cauchy problem for any supercritical initial velocity inside the light sphere is continued uniquely; it is given by the separatrix OB. For $l > 5/4$, the continuation of the solution for $r < ct$ becomes nonunique, but necessarily with shock waves.

2) In the case of axial symmetry, for $2/3 < l \leq 5/6$, solutions with an initial expansion velocity from the axis equal to the speed of light are realized, and only they. When $l \in (2/3, 3/4)$, the only continuation of the solution $V = 1$ inside the light sphere is the curve OB. The choice of continuation inside the light cylinder, if $l \in (3/4, 5/6)$, is nonunique; however, for all continuations shock waves arise through $\lambda = 1$. In particular, through the singular cylinder $r = ct$ the solution $V = 1$ can be continued analytically up to some $\tilde{\lambda}$, at which the velocity decreases by a jump from the speed of light to the velocity $c(3\tilde{\lambda}+1)/(\tilde{\lambda}+3)$. If $l > 5/6$, then there exists a certain critical initial velocity. Only motions with initial velocities not less than the critical one are physically realizable. The continuation of the solution inside the light cylinder becomes unique, but not analytic: for $r < ct$ the matter moves with the speed of light up to some $r = ct\tilde{\lambda}$, at which the velocity drops by a jump to the value $c(3\tilde{\lambda}+1)/(\tilde{\lambda}+3)$.

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Note: Figure translations are in progress. See original paper for figures.