ON THE SUMMABILITY (BOUNDEDNESS) OF SOLUTIONS OF ONE CLASS OF HYPOELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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Abstract

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ON THE SUMMABILITY (BOUNDEDNESS) OF SOLUTIONS OF ONE CLASS OF HYPOELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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In the present communication we continue the investigation, begun by the author in (1), of solutions of the equation

$$L(\partial/\partial x)U = f, \tag{1}$$

We shall assume below that the characteristic polynomial $L(is)$, with complex constant coefficients, has no real zeros for $|s| \neq 0$ and is quasi-homogeneous, i.e., for every $\lambda > 0$ the equality

$$P(is\lambda) = \lambda P(is)$$

holds.

The vector $\alpha = (a_1, \dots, a_n)$ will be called the homogeneity exponent of the operator. If $f \in L_p(C_\varepsilon^l)$, then, as was shown in ([1]), equation (1) is always solvable in classes of functions that consist of summable (bounded) derivatives of order ρ, where $\rho \cdot \alpha = 1$. Solvability of equation (1) in narrower classes of functions containing summable (bounded) derivatives of lower orders, generally speaking, does not occur. However, one can distinguish a class of regular solutions for which, under certain conditions, one can guarantee the summability (boundedness) of derivatives of all orders ρ, where $0 \leq \rho \leq 1$. These conditions are formulated below in the form of four theorems.

This work is close in subject matter to the works ([11, 2–4]), where other conditions are imposed on the operator $L(\partial/\partial x)$, the most essential of which is the absence of real zeros of $L(is)$ in ([2]), and of common zeros of $L(is)$ and $\operatorname{grad} L(is)$ in ([3,4]). The results obtained also develop the well-known works of S. M. Nikol' skii ([5]), V. P. Il' in ([6]) on estimates of partial derivatives of functions of many variables, and the related works ([7,8]) on coercivity.

Let us introduce definitions. Denote by R_n the Euclidean space of points

$$x = (x_1 \dots x_n); \quad x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}; \quad |x| = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2};$$

$$D^\beta = D^{\beta_1} \dots D^{\beta_n} = \partial^\beta / \partial x_1^{\beta_1} \dots \partial^{\beta_n} / \partial x_n^{\beta_n}; \quad |l| = \sum_{i=1}^{n} l_i; \quad l \cdot \alpha = \sum_{i=1}^{n} l_i \alpha_i;$$

$$\Delta(f, hl_i) = f(x_1, \dots, x_i, \dots, x_n) - f(x_1, \dots, x_i + h, \dots, x_n),$$

$$R_v[f(x)] = \prod_{i=1}^{n} v_i^{-1} \int_0^{v_1} \cdots \int_0^{v_n} f(x+u)\, du; \tag{2}$$

$$\|f, L_p^l\| = \sum_{i=1}^{n} \|D^i f, L_p\|, \qquad 1 < p < \infty; \tag{3}$$

$$\|f, L_p^{l,\alpha}\| = \sum_{\rho \cdot \alpha = 1} \|D^\rho f, L_p\| + \|D^\theta f, L_p^l\|; \tag{4}$$

$$\|f, L_{1,\gamma}\| = \int_{R_\nu} \prod_{i=1}^{\nu} (1 + |x_i|)^{\gamma_i} |f(x_1, \dots, x_n)|\, dR_\nu, \qquad \gamma = (\gamma_1, \dots, \gamma_\nu),$$

$$1 \le \nu \le n; \tag{5}$$

$$\|f, C_\varepsilon, L_{1,\gamma}\| = \sup_{|v| \ge 0,\ x \in R_n} \left| v^\varepsilon R_\nu [\|f, L_{1,\gamma}\|] \right|; \tag{6}$$

$$\|f, C_\varepsilon\| = \sup_{|v| \ge 0,\ x \in R_{n-\nu}} \left| v^\varepsilon R_\nu [f(x)] \right|; \tag{7}$$

$$\varepsilon = (\varepsilon_{\nu+1}, \dots, \varepsilon_n), \qquad \gamma = (\gamma_1, \dots, \gamma_\nu), \qquad 0 \le \nu \le n-1.$$

For $p = \infty$, l an integer, $0 < \mu < 1$, put

$$\|f, C^{l+\mu}\| = \sum_{i=1}^{n} \left(\sup_{|h| \le 1,\ x \in R_n} \left| \frac{\Delta(f_i^{(l_i)}, hl_i)}{h^{\mu_i}} \right| + \sum_{k=0}^{l_i} |D^k f| \right); \tag{8}$$

$$\|f, C_\varepsilon^{l+\mu}\| = \sum_{i=1}^{n} \|f, C^{l+\mu}\| + \sum_{k=0}^{l_i} \|f^k, C_\varepsilon\|; \tag{9}$$

$$\|f, C_\varepsilon^{l+\mu,\alpha}\| = \sum_{\rho\cdot\alpha=1} \|D^\rho f, C_\varepsilon^{l+\mu}\|. \tag{10}$$

We shall say that a function f belongs to the functional space $L_p^l(C_\varepsilon^l)$ if f belongs to the closure, in the corresponding norm, of smooth functions all of whose derivatives are summable (for which the norm is finite). Define a generalized solution of equation (1) as the limit, in the norm (4) (respectively, (10)), of solutions of the equation

$$L(\partial/\partial x)U_\nu = f_\nu,$$

where f_ν converges to f in the norm $L_p^l(C_\varepsilon^l)$.

We shall call a solution U of equation (1) regular if on every compact $G \in R_n$, $U \in W_p^{1/\alpha}(G)$ $(C^{1/\alpha+\mu})$, and there exists a natural number N such that for almost all $x = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ one has

$$\lim_{|x_i|\to\infty} (1 + |x_i|)^N |D^\rho U| = 0 \tag{11}$$

for $i = 1, \dots, n$, $0 \leq \rho \cdot \alpha \leq 1$. Put

$$G_0(x) = \prod_{i=1}^{n} G_i(x_i) = \prod_{i=1}^{n} \exp[-(m_i\alpha_i)^{-1} x_i^{m_i}], \tag{12}$$

where the m_i are sufficiently large numbers,

$$G(x) = G_0(x) \sum_{i=1}^{n} x_i^{m_i} [L(ix)]^{-1}. \tag{13}$$

Then the following theorem on the representation of functions holds, generalizing the corresponding Radon formula ([9]).

Theorem 1. Let U be summable on every compact $G \subset R_n$, and let

$$\lim_{|v|\to\infty} R_\nu[U] = 0$$

for almost all $x \in R_n$.

Then for almost all $x \in R_n$,

$$U = \lim_{h\to 0} U_h = \lim_{h\to 0} \frac{L(\partial/\partial x)}{(2\pi)^n} \int_h^{1/h} v^{-|\alpha|}\, dv \int_{R_n} \widehat{G}\Big(\frac{t-x}{v^\alpha}\Big)\, U(t)\, dt. \qquad (14)$$

Corollary. Let $f \in L_p(C^l_\varepsilon)$, and let U be a regular solution of (1) satisfying the condition

$$\lim_{|v|\to\infty} R_\nu[D^\rho U] = 0 \qquad \text{for almost all } x \in R_n \text{ (for all } x). \qquad (15)$$

Then for almost all $x \in R_n$ (for all x) one has

$$D^\rho U = \lim_{h\to 0} \frac{1}{(2\pi)^h} \int_h^{1/h} v^{-|\alpha|-\rho\alpha}\, dv \int_{R_n} D^\rho \widehat{G}\Big(\frac{t-x}{v^\alpha}\Big)\, f(t)\, dt. \qquad (16)$$

Using Theorem 1, one can obtain the following assertions.

Theorem 2. Let $f \in C^{l+\mu,\alpha}_\varepsilon$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $0 < \varepsilon_i \le 1$, $\rho = (\rho_1, \dots, \rho_n)$, $\rho_i \ge 0$, $i = 1, \dots, n$, $0 < \mu_i < 1$, $l_i > 0$, $\mu_i = \mu\alpha_i^{-1}$. If $(\rho + \varepsilon)\alpha > 1$, $\rho\alpha < 1$, then there exists a solution of equation (1) $U \in C^{l+\mu,\alpha}_\varepsilon$, whose partial derivative $D^\rho U \in C$, and the estimate holds

$$C_1 \|f, C^{l+\mu}_\varepsilon\| \le \|U, C^{l+\mu,\alpha}_\varepsilon\| + \|D^\rho U, C^{l+\mu}_\varepsilon\| \le C_2 \|f, C^{l+\mu}_\varepsilon\|, \qquad (17)$$

where $\bar\varepsilon_i < \varepsilon_i(1 - (1-\rho\alpha)/|\alpha|)$, $i = 1, \dots, n$. Every regular solution v satisfying condition (15) has estimate (17).

Corollary. Let $f \in L_p(R_n) \cap C^{l+\mu}(R_n)$, $1 < p < \infty$, $l_i > 0$, $0 < \mu_i < 1$; if $|\alpha| > p$, $\mu_i = \mu\alpha_i^{-1}$, then equation (1) has a solution

$$U_0 \in C^{l+\mu+1/\alpha}_{\bar\varepsilon}, \quad C_1\|f, C^{l+\mu}_\varepsilon\| \le \|U_0, C^{l+\mu+1/\alpha}_{\bar\varepsilon}\| \le C_2(\|f, L_p\| + \|f, C^{l+\mu}\|),$$

where

$$\bar\varepsilon_i < \frac{1}{p}\left(\frac{|\alpha|-1}{|\alpha|}\right), \qquad i = 1, 2, \dots, n.$$

Every regular solution v satisfying condition (15) for $|\rho| = 0$ coincides with U_0.

The following assertion generalizes to the case of operators the known Sobolev embedding theorem ([10]) for the limiting exponent (see also ([5,6])).

Theorem 3. Let $f \in L_p(R_n)$, $1 < p < \infty$, $p < q < \infty$, $\varkappa = \rho\alpha + (1/p - 1/q)|\alpha| = 1$.

Then there exists a solution U_0 of equation (1), for which the estimate holds

$$C_1\|f, L_p\| \leq \|U_0, L_p^{l,\alpha}\| + \|D^\rho U_0, L_q\| \leq C_2\|f, L_p\|. \tag{18}$$

Every regular solution v, for which (15) holds, satisfies estimate (18).*

Remark. From the embedding theorem for the limiting exponent it follows that Theorem 2 is sharp in the limiting sense.

Theorem 4. Let $f \in L_p \cap L_r$, $1 < p \leq r < q < \infty$ and, moreover, $\varkappa_1 = \rho\alpha + (1/p - 1/q)|\alpha| \geq 1$, $\varkappa_2 = \rho\alpha + (1/r - 1/q)|\alpha| \leq 1$.

Then equation (1) has a solution U_0, for which the estimate holds

$$C_1(\|f, L_p\| + \|f, L_r\|) \leq \|U_0, L_p^{l,\alpha}\| + \|U_0, L_r^{l,\alpha}\| + \|D^\rho U_0, L_q\| \leq C_2(\|f, L_p\| + \|f, L_r\|). \tag{19}$$

For every regular solution v satisfying (15), estimate (19) is valid.

Theorem 2 can be strengthened if certain structural requirements are imposed on the right-hand side of equation (1) (oddness, equality to zero of certain integrals, etc.). The following Theorem 5 gives some idea of the character of the behavior of solutions in this case.

Theorem 5. Let $f = C^{l+\mu}$, and let the norms be finite

$$\|f, C_\varepsilon, L_{1,\gamma}\| < \infty, \qquad \|f, C_{\hat{\varepsilon}}\| < \infty,$$

$$\varepsilon = (\varepsilon_{\nu+1}, \dots, \varepsilon_n), \qquad \gamma = (\gamma_1, \dots, \gamma_\nu), \qquad \hat{\varepsilon} = (\hat{\varepsilon}_1, \dots, \hat{\varepsilon}_n),$$

$$0 < \varepsilon_i \leq 1, \qquad \gamma_i > 1, \qquad 0 < \hat{\varepsilon}_i \leq 1, \qquad i = 1, 2, \dots, n,$$

and, moreover, for each i, $1 \leq i \leq \nu$, it holds that

$$f(x_1, \dots, -x_i, \dots, x_n) = -f(x_1, \dots, x_i, \dots, x_n).$$

* Coercive estimates of the form (18) were obtained earlier by O. V. Besov ([8]).

Then, if

$$\varkappa_1 = \rho \cdot \alpha + \sum_{i=\nu+1}^{n} \varepsilon_i a_i + 2\sum_{i=1}^{n} a_i > 1, \qquad \varkappa_2 = (\rho + \hat{\varepsilon})\alpha < 1,$$

then there exists a solution U_0 of equation (1) for which

$$C_1\|f,\ C_{\bar\varepsilon}^{l+\mu}\| \le \|U_0,\ C_{\bar\varepsilon}^{l+\mu,\alpha}\| + \|D^\rho U_0,\ C\| \le$$

$$\le C_2(\|f,\ C_{\bar\varepsilon} L_{1,\gamma}\| + \|f,\ C_\varepsilon\| + \|f,\ C^{l+\mu}\|), \tag{20}$$

$$\bar\varepsilon_i < \begin{cases} 1, & i = 1, \dots, \nu, \\ \varepsilon_i, & i = \nu + 1, \dots, n. \end{cases}$$

For any regular solution v of equation (1) satisfying (15), the estimate (20) holds.

Corollary. Let $f \in C^{l+\mu}$, $\|f, L_{1,\gamma}\| < \infty$, $\|f, C_\varepsilon\| < \infty$, $\gamma = (\gamma_1, \dots, \gamma_n)$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\gamma_i > 1$, $0 < \varepsilon_i \le 1$, $i = 1, 2, \dots, n$,

$$\varkappa_1 = |\alpha| + \rho\alpha + \min_{1 \le i \le n} a_i > 1, \qquad \varkappa_2 = (\varepsilon + \rho)\alpha < 1.$$

Then, if $\int_{R_n} f\, dR_n = 0$, there exists a solution U_0 of equation (1) satisfying the estimate

$$C_2\|f,\ C_{\bar\varepsilon}^{l+\mu}\| \le \|U_0,\ C_{\bar\varepsilon}^{l+\mu,\alpha}\| + \|D^\rho U_0,\ C\| \le$$

$$\le C_1(\|f, L_{1,\gamma}\| + \|f, C^{l+\mu}\| + \|f, C_\varepsilon\|), \qquad \bar\varepsilon_i < 1, \quad i = 1, 2, \dots, n. \tag{21}$$

Every regular solution v of equation (1) satisfies condition (21), if condition (15) is fulfilled.

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* In the author' s work (1), an inaccuracy was allowed in the formulation of Theorem 4. In the condition of the theorem the following additional conditions should be introduced: $l = (l_1, \dots, l_n)$, $l_i = \bar{l}_i + \mu_i$, $\bar{l}_i$ an integer, $0 < \mu_i < 1$, $\mu_i = \mu a_i^{-1}$, where $a = (a_1 \dots a_n)$ is the homogeneity exponent of the operator $L(D)$.

Note: Figure translations are in progress. See original paper for figures.