THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION IN A COMPLEX DOMAIN

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Abstract

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MATHEMATICS

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THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION IN A COMPLEX DOMAIN

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1. The purpose of the present note is to obtain an integral representation of the solution of the Cauchy problem for the Laplace equation, analogous to Kirchhoff' s formula for the wave equation. Such a representation is obtained after analytic continuation of the equation and of the Cauchy data into the complex domain. For the case of two independent variables, such a representation ("Riemann' s formula") was obtained in ([1]). Our considerations are adjacent to fundamental investigations ([2–4]). For simplicity of exposition, we shall consider here only the 3- and 4-dimensional cases. The Cauchy problem for the Laplace equation with analytic Cauchy data arises naturally, in particular, in the mathematical study of electrochemical machining of metal ([5–6]), where it is a quite direct and adequate description.

2. We want to obtain Kirchhoff' s formula for the solution of the equation

$$\Delta u \equiv \sum_{j=0}^{n} \frac{\partial^2 u}{\partial x_j^2} = f(x_0, \dots, x_n) \quad (n = 2, 3), \tag{1}$$

satisfying on the surface $\Gamma : x_0 = F(x_1, \dots, x_n)$ the Cauchy conditions

$$u|_\Gamma = \varphi(x_1, \dots, x_n), \qquad \partial u/\partial x_0 = \psi(x_1, \dots, x_n), \tag{2}$$

where $f(x)$, $F(x)$, $\varphi(x)$, $\psi(x)$ are analytic functions of the real variables $x = (x_1, \dots, x_n)$. Continue problem (1)–(2) into the complex domain $z_j = x_j + iy_j$ $(j = 0, 1, \dots, n)$ and seek there a holomorphic, in z_j, solution $u(z)$ of the problem

$$\sum_{j=0}^{n} \frac{\partial^2 u}{\partial z_j^2} = f(z), \tag{3}$$

$$u|_\Gamma = \varphi(z), \qquad \partial u/\partial z_0|_\Gamma = \psi(z), \tag{4}$$

where Γ is now already a complex surface, i.e. a $2n$-dimensional manifold in the $2(n+1)$-dimensional z-space C^{n+1}. For simplicity of exposition we shall suppose that $F(0) < 0$, $F_{,i}(0) = 0$ (this can always be achieved by a real orthogonal transformation not changing (1)), and restrict ourselves to finding the solution at the origin, which we assume to be sufficiently close to Γ (the solution exists only locally). Consider the characteristic cone K with vertex at zero, $K : \{\sum_{j=0}^{n} z_j^2 = 0\}$. Denote by B the intersection of the cone K with the Cauchy surface Γ:

$$B = K \cap \Gamma = \{z_0 = F(z_1, \dots, z_n),\ z_1^2 + \dots + z_n^2 = -F^2(z_1, \dots, z_n) = -F^2(0) + F(0)O(z^2)\}.$$

Alongside B, consider the $(n-1)$-dimensional complex sphere

$$CS^{n-1} = \bar{B} : \quad \{z_0 = F(0),\ z_1^2 + \dots + z_n^2 = -F^2(0)\},$$

which is the intersection of K with the plane (complex) $z_0 = F(0)$. The complex sphere CS^{n-1} is the tangent bundle TS^{n-1} of the real sphere S^{n-1} and therefore admits a fibration $CS^{n-1} \to S^{n-1}$ with base S^{n-1} and fiber

R^{n-1}. Hence it follows that the $(n-1)$-dimensional group of compact homologies $H_c^{n-1}(CS^{n-1}) = Z_0$. The sphere $\widetilde{B}_c = \{x_1 = \dots = x_n = 0;\ y_1^2 + \dots + y_n^2 = F^2(0)\} \subset \widetilde{B}$ is a generating cycle for $H_c^{n-1}(\widetilde{B})$. By virtue of our assumptions on Γ, there exists in B an $(n-1)$-dimensional cycle B_c, diffeomorphic to the cycle $\widetilde{B}_c$ (a vanishing Picard–Lefschetz cycle [4]). Denote by $K_c \subset K$ the compact submanifold of K over K_c passing through the vertex (for example, the cone over B_c); by $\overline{B}_c$ the submanifold (with boundary) on Γ spanning B_c: $(b\overline{B}_c = B_c *)$; and by $\overline{K}_c$ the manifold spanning K_c

$$(b\overline{K}_c = K_c \cup \overline{B}_c).$$

Let, further,

$$r = \left(\sum_{j=1}^{n} z_j^2\right)^{1/2} \quad \text{and} \quad \rho = \left(\sum_{j=0}^{n} z_j^2\right)^{1/2}$$

be certain branches of the corresponding multivalued functions on K_c and on $\overline{K}_c$. Then the Kirchhoff formulas hold:

$$4\pi u(0) = \int_{K_c} * \frac{f'}{r} + \int_{B_c} * \left(\frac{\varphi}{r^2}\, dr + d\varphi\, \frac{1}{r} - \frac{i\psi}{r}\, dr\right). \qquad (5)$$

Here $n = 3$; $*$ is the metric operator (formal) in Euclidean space z_1, z_2, z_3,

$$ds^2 = \sum_{j=1}^{3} dz_j^2, \qquad f'(z_1, z_2, z_3) = f(ir, z_1, z_2, z_3),$$

and

$$2\pi u(0) = \int_{\overline{K}_c'} \frac{f(z_0, z_1, z_2)}{\rho} \, dz_0 \, dz_1 \, dz_2 + \int_{\overline{B}_c'} \frac{1}{\rho} \left[-\frac{\varphi}{z_0^3} \, d\left(\frac{z_0^2}{2} * \frac{dr^2}{2} \right) + d\varphi * \left(dz_0 + \frac{1}{z_0} \frac{dr^2}{2} \right) + \frac{\psi}{z_0^2} \, d\left(\frac{z_0^2}{2} * \frac{dr^2}{2} \right) \right] \tag{6}$$

$n = 2$, $*$ is in the Euclidean space z_1, z_2;

$$ds^2 = \sum_{j=1}^{2} dz_j^2.$$

For comparison with the real wave case, we note that, in our notation, the classical Kirchhoff formula has the form ($n = 3$)

$$4\pi u(0) = \int_K * \frac{f'}{r} + \int_B * \left(-\frac{\varphi}{r^2} \, dr - d\varphi \, \frac{1}{r} + \frac{\psi}{r} \, dr \right).$$

A direct proof of formulas (5)–(6) can be given, for example, by Hadamard' s method of finite parts ([7]), or by using the corresponding technique of generalized functions ([8]). We shall give, however, a more usual derivation: for $n = 3$, considering (3) on K (and then on K_c), we obtain formula (5), and from it (6) by the descent method.

3. In deriving (5) we shall follow the exposition ([9]) of the corresponding derivation for the wave equation. Introduce new variables

$$z_0' = z_0 + ir, \qquad z_j' = z_j \quad (j = 1, 2, 3),$$

where

$$r = \left(\sum_{j=1}^{3} z_j^2 \right)^{1/2}$$

is a two-valued function on K with branch manifold

$$\sum_{j=1}^{3} z_j^2 = 0.$$

* The symbol bA everywhere denotes the boundary of A.

** A prime in the notation of a geometric object means that the object is taken in the three-dimensional problem; for example, $\dim \overline{K}_c' = \dim K_c = 3$, but $\dim K_c' = 2$.

Near K_c one can single out a single-valued branch r, taking values in the lower half-plane on $\widetilde{K}_c$: $y_0 = 0$, $x_j = 0$ $(j = 1, 2, 3)$,

$$\sum_{j=1}^{3} y_j^2 - x_0^2 = 0, \qquad x_0 < 0.$$

Equation (3) in the new variables has the form

$$Lu \equiv \sum_{j=1}^{3} \frac{\partial^2 u}{\partial z_j'^2} + 2i\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial z_0'}\right) \equiv \Delta u + l u_{z_0'} = f. \tag{7}$$

Here

$$\frac{\partial}{\partial r} = \sum_{j=1}^{3} \frac{z_j'}{r}\frac{\partial}{\partial z_j'},$$

Δ is the Laplace operator in z_j' $(j = 1, 2, 3)$; $lv = 2ir * d(*dr \cdot v/r)$. On the cone K we shall regard u and $u_{z_0'}$ as independent functions; the bilinear Green formula holds $(l^* = -l)$:

$$*v(\Delta u + l u_{z_0'}) - [*u \cdot \Delta v - *u_{z_0'} lv]$$

$$= d\,[u * dv - v * du + 2i * dr \cdot u_{z_0'} v]. \tag{8}$$

The function $v = 1/r$ satisfies the equations $\Delta v = 0$ and $lv = 0$. Integrating the n-forms (8) for $v = 1/r$ over K_c, we obtain, analogously to the real case,

$$\int_{K_c} f * \frac{1}{r} = 4\pi u(0) - \int_{B_c} u * d\frac{1}{r} - *du \cdot \frac{1}{r} + 2i\frac{1}{r} * dr \cdot u_{z_0'},$$

and, passing to the old variables, formula (5).

4. Suppose that in (5) the functions F, f, φ, ψ do not depend on z_3. Then K_c and B_c can be obtained from $\widetilde{K}_c'$ and $\widetilde{B}_c'$ by lifting the latter from the subspace z_0, z_1, z_2, $(z_3 = 0)$ by the formula $z_3 = \pm\sqrt{z_0^2 + z_1^2 + z_2^2} = \pm i\rho$. Passing in (5), by means of this formula, to the independent variables z_0, z_1, z_2, varying in $\widetilde{K}_c'$ $(\widetilde{B}_c')$, we obtain (6).

5. Our constructions can be carried out, generally speaking, only locally: at large distances from Γ the characteristic cone will begin to touch Γ, as in the real wave case. This is connected, generally speaking, with the multivalued nature of the solution and the need to consider it not in C^{n+1}, but in the corresponding Riemann space.

6. In the special case $F(z_1, \dots, z_n) = \text{const}$, our constructions can be carried out in the space $y_0 = 0$, $x_1 = \dots = x_n = 0$; then (5) and (6) turn into the classical Kirchhoff formulas for the equation

$$\frac{\partial^2 u}{\partial x_0^2} - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial y_j^2} = f$$

with initial data for $x_0 = \text{const}$. The problem is globally solvable.

7. Our considerations show that the ill-posedness of the Cauchy problem for the Laplace equation in a real domain formally disappears in the complex one: under a small change of the initial data in the corresponding metric (analogously to the real wave case–see ([10])), the solution changes little. Of course, the analytic character of the initial data makes this assertion rather meager in content.

8. The case of equation (1) for $n = 2m + 1$ can be treated analogously to the case $n = 3$, replacing the form $\omega_0 = *1/r$ by the form ω_{m-1} (see ([8])), while the case $n = 2m$ is obtained from the preceding one by the method of descent. For any nondegenerate equation of the 2nd order in n-dimensional space

with analytic coefficients can be carried out analogously, for example by using the technique developed by S. L. Sobolev for equations of normal hyperbolic type ((10), §§ 19, 20).

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Note: Figure translations are in progress. See original paper for figures.