ON A TAUBERIAN THEOREM OF M. V. KELDYSH

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Abstract

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MATHEMATICS

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ON A TAUBERIAN THEOREM OF M. V. KELDYSH

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Let $\psi(u)$ and $\varphi(u)$ be positive increasing functions defined for $u > 0$, with $\varphi(u)$ differentiable and satisfying the conditions: $\lim\limits_{u\to\infty} \varphi(u) = \infty$ and, as $u \to 0$,

$$\alpha\varphi(u) < u\varphi'(u) < \beta\varphi(u),$$

where α and β are positive constants satisfying $0 < \beta < \alpha + 1$, and, for $u < a/2$, $\varphi(u) = 0$, $\psi(u) = 0$, and let

$$\Phi_1(x) = \int_0^\infty \frac{d\varphi(u)}{(u+x)^m}, \qquad \Phi_2(x) = \int_0^\infty \frac{d\psi(u)}{(u+x)^m} \quad (m > \beta + 1).$$

If $\Phi_2(x) \sim \Phi_1(x)$ as $x \to \infty$, then $\psi(x) \sim \varphi(x)$ as $x \to \infty$. (1)

This theorem belongs to M. V. Keldysh (^{1}) and has an important application in spectral theory. Subsequently, in (^{2}) the conditions of this theorem were somewhat weakened, and in (^{3}) the remainder term was calculated. In particular, in (^{3}) it was shown that if, instead of (1), one requires the condition

$$\Phi_2(x) = \Phi_1(x)\{1 + O[r(x)]\}, \qquad x \to \infty,$$

then

$$\psi(x) = \varphi(x)\left\{1 + O\left(\frac{1}{\ln x}\right)\right\}, \qquad \text{if } r(x) = x^{-\gamma},\ \gamma > 0, \tag{2}$$

$$\psi(x) = \varphi(x)\{1 + O(x^{-\delta})\} + O(1)^*, \qquad \text{if } r(x) = e^{-cx^\delta},\ c > 0,\ 0 < \delta \leq \tfrac{1}{2}. \tag{3}$$

Under the given conditions these estimates cannot be improved (^{4}). However, in solving concrete problems we may have more information about the functions $\Phi_1(x)$ and $\Phi_2(x)$ and obtain better results than in (2) and (3), even under weaker requirements imposed on $\varphi(x)$.

In this note we shall give several results of this kind.

Here and below, the logarithm means the principal branch of the logarithmic function, and l is the curve $|y| = c_0(\varepsilon - x)^\tau$, $c_0 > 0$, $0 \leq \tau < 1$, $x \leq \varepsilon$; ε

is a sufficiently small positive number. For $\tau = 0$, by l we shall mean the two half-lines $y \pm c_0$, joined by the segment of the straight line $x = \varepsilon$.

Theorem 1. Let $\psi(u)$ and $\varphi(u)$ be positive increasing functions defined for $u \geq 0$, and let $\psi(u) = \varphi(u) = 0$ if $0 < u < h$, $h = \text{const}$. Suppose, moreover, that the integrals

$$\Phi_1(z) = \int_0^\infty \frac{\varphi(u)\,du}{u^\mu (u+z)^{\nu+1}}, \qquad \Phi_2(z) = \int_0^\infty \frac{\psi(u)\,du}{u^\mu (u+z)^{\nu+1}}, \qquad \mu \geq 0,\ \nu \geq 0,$$

converge for $z \in l$ and satisfy the condition

$$\Phi_2(z) = \Phi_1(z) + O(|z|^{-\omega}), \qquad 0 < \omega < 1,\ z \in l,\ z \to \infty. \tag{4}$$

Then

$$\psi(x) = \varphi(x) + O\left(\frac{\varphi(x)}{x^{1-\tau}}\right) + O(x^{\mu+\nu-\omega}), \qquad x \to \infty.$$

* Here $O(1)$ may be replaced by a constant.

Theorem 2. Let $\psi(u)$ and $\varphi(u)$ satisfy the conditions of Theorem 1, and let the integrals

$$F_1(z) = \int_0^\infty \frac{d\varphi(u)}{u^\mu (u+z)^{\nu+1}}, \qquad F_2(z) = \int_0^\infty \frac{d\psi(u)}{u^\mu (u+z)^{\nu+1}}, \qquad \mu \geq 0, \quad \nu \geq 0,$$

converge for $z \in l$ and satisfy the condition

$$F_2(z) = F_1(z) + O(|z|^{-\omega}), \qquad 0 < \omega < 1,\ z \in l,\ z \to \infty.$$

Then

$$\psi(x) = \varphi(x) + O\left(\frac{\varphi(x)}{x^{1-\tau}}\right) + O(x^{\mu+\nu+1-\omega}), \qquad x \to \infty.$$

Theorem 3. Let $\psi(u)$ satisfy the conditions of Theorem 1, and let the integral

$$\Phi(z) = \int_0^\infty \frac{z^{p+1}\psi(u)}{u^{p+1}(u+z)}\,du, \qquad p > 0,$$

converge for $z \in l$ and satisfy the condition

$$\Phi(z) = z^p L(z) + O(|z|^{\rho-\omega}), \qquad \rho > 0,\ 0 < \omega < 1,\ z \in l,\ z \to \infty,$$

where $L(z)$ is regular in the domain $|y| \geq c_0(\varepsilon - x)^\tau$. Then

$$\psi(x) = \frac{x^p}{2\pi i}\Delta_{\Gamma_1}\{z^{\rho-p}L(z)\} - \frac{x^{\tau+p}}{\pi}\operatorname{Re}\left\{\frac{d}{dz}\left[z^{\rho-p}L(z)\right]\right\}$$

$$+O\left\{x^{\tau+p}\left|\operatorname{Im}\left[z^{-p}\frac{d}{dz}(z^\rho L(z))\right]\right|\right\} + O(x^{\rho-\omega}), \qquad x \to \infty.$$

Here $\Delta_{\Gamma_1}\{z^{\rho-p}L(z)\}$ denotes the increment of the function $z^{\rho-p}L(z)$ with respect to z along the arc Γ_1 of the curve l_1, $|y| = 2c_0(\varepsilon - x)^\tau$, lying between the points $\bar{z} = -x - iy$ and $z = -x + iy$.

Theorem 4. Let

$$0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots; \qquad \lim_{n\to\infty} \lambda_n = \infty;$$

let $a_n(t)$ be real functions; and let the series

$$G(t, \tau; z) = \sum_{n=0}^{\infty} \frac{a_n(t)a_n(\tau)}{\lambda_n + z}$$

converge for $z \in l$ and, uniformly with respect to t and τ from some bounded domain D, satisfy the condition

$$G(t, \tau; z) = F(z, \rho) + O(|z|^{-\omega}), \qquad z \in l,\ z \to \infty,$$

where ρ is the distance between the points t and τ. Suppose further that

$$|F(z, \rho)| \leq CF(x, \rho), \qquad z \in \Gamma, \qquad \lim_{\rho\to\infty} F(z, \rho) = Hz^{-\gamma}, \qquad 0 < \gamma < \omega,$$

where C, H are constants. Then

$$\sum_{\lambda_n \leq x} a_n(t)a_n(\tau) = \Phi(x, \rho) + O(F(x, \rho)x^\tau) + O(x^{\tau-\gamma}) + O(x^{1-\omega}), \qquad x \to \infty.$$

Here

$$\Phi(x, \rho) = \frac{1}{2\pi i}\int_\Gamma F(z, \rho)\, dz,$$

and Γ is the arc of the curve l lying between the points $\bar{z} = -x - iy$ and $z = -x + iy$.

All the theorems are proved by one method.

We give the main points of the proof of Theorem 1.

Using the device employed in the paper (1), from condition (4) we obtain

$$\int_0^\infty \frac{\psi(u) - \varphi(u)}{u^{\mu+\nu}(u+z)}\, du = O(|z|^{-\omega}), \qquad \text{if } z \in l,\ z \to \infty. \tag{5}$$

Let

$$R(z) = \int_0^\infty \frac{z^{\mu+\nu}[\psi(u) - \varphi(u)]}{u^{\mu+\nu}(u+z)}\, du. \tag{6}$$

According to (4), for $z \in l$ we shall have

$$\left| \frac{R(z)}{z^{\mu+\nu-\omega}} \right| < M \qquad (M = \text{const}).$$

Moreover, since in the domain E, defined by the condition $|y| \geq c_0(\varepsilon - x)^\tau$, the function $R(z)/z^{\mu+\nu-\omega}$ is regular and, for any $\delta > 0$,

$$R(z)/z^{\mu+\nu-\omega} = O(e^{r^\delta}), \qquad r = |z| \to \infty,$$

uniformly in E, application of the Phragmén-Lindelöf theorem leads to the estimate

$$R(z) = O\left(|z|^{\mu+\nu-\omega}\right), \qquad z \to \infty, \tag{7}$$

if z belongs to the domain E. Hence, applying Cauchy' s theorem, we obtain

$$R'(z) = O\left(|z|^{\mu+\nu-\omega-\tau}\right), \qquad z \to \infty, \tag{8}$$

if $z \in l_1$, where l_1 is the curve $|y| = 2c_0(\varepsilon - x)^\tau$.

On the other hand, taking into account (5), (8), and the fact that $\psi(u)/u^{\mu+\nu} \to 0$, $\varphi(u)/u^{\mu+\nu} \to 0$ as $u \to \infty$, we obtain

$$R'(z) = \int_0^\infty \frac{z^{\mu+\nu-1}\, d[\psi(u) - \varphi(u)]}{u^{\mu+\nu}(u+z)}. \tag{9}$$

From (5), (8), (9) we have

$$y^2\int_0^\infty \frac{\psi(u)\,du}{u^{\mu+\nu}[(u-x)^2+y^2]} = y^2\int_0^\infty \frac{\varphi(u)\,du}{u^{\mu+\nu}[(u-x)^2+y^2]} + O(x^{\tau-\omega}), \qquad x\to\infty, \tag{10}$$

$$y^2\int_0^\infty \frac{d\psi(u)}{u^{\mu+\nu-1}[(u-x)^2+y^2]} = y^2\int_0^\infty \frac{d\varphi(u)}{u^{\mu+\nu-1}[(u-x)^2+y^2]} + O(x^{1-\omega}), \qquad x\to\infty. \tag{11}$$

From (5), (8)-(11) it follows that

$$y^2\int_0^\infty \frac{\varphi(u)\,du}{u^{\mu+\nu}[(u-x)^2+y^2]} = O\left(\frac{\varphi(x)}{x^{\mu+\nu-\tau}}\right) + O\left(\frac{1}{x^{2-2\tau}}\right), \qquad x\to\infty; \tag{12}$$

$$y^2\int_0^\infty \frac{d\varphi(u)}{u^{\mu+\nu-1}[(u-x)^2+y^2]} = O\left(\frac{\varphi(x)}{x^{\mu+\nu-\tau}}\right) + O\left(\frac{1}{x^{2-2\tau}}\right), \qquad x\to\infty. \tag{13}$$

Now let x be a sufficiently large number and let Γ_1 be the arc of the curve l_1 enclosed between the points $\bar{z}=-x-iy$ and $z=-x+iy$. Then, according to (6)-(9), we shall have

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{\Gamma_1}\left\{\int_0^\infty \frac{d[\psi(u)-\varphi(u)]}{u^{\mu+\nu-1}(u+z)} - (\mu+\nu-1)\int_0^\infty \frac{\psi(u)-\varphi(u)}{u^{\mu+\nu}(u+z)}\,du\right\}dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_1}\frac{d}{dz}\left[z^{-\mu-\nu+1}R(z)\right]dz = O(x^{1-\omega}), \qquad x\to\infty. \end{aligned} \tag{14}$$

On the other hand, since

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{dz}{u+z} = \begin{cases} 1+\dfrac{y(u-x)}{\pi[(u-x)^2+y^2]} + O\left(\dfrac{y^2}{(u-x)^2+y^2}\right), & u\le x, \\ \dfrac{y(u-x)}{\pi[(u-x)^2+y^2]} + O\left(\dfrac{y^2}{(u-x)^2+y^2}\right), & u>x, \end{cases}$$

then, by virtue of (6)–(13), we find

$$\begin{aligned} I &= \int_0^x \frac{d[\psi(u)-\varphi(u)]}{u^{\mu+\nu-1}} - (\mu+\nu-1)\int_0^x \frac{\psi(u)-\varphi(u)}{u^{\mu+\nu}}\,du + O(x^{\tau-\omega}) + O(x^{1-\omega}) \\ &\quad + O(x^{2\tau-2}) + O(x^{-\mu-\nu+\tau}) = \frac{\psi(x)-\varphi(x)}{x^{\mu+\nu-1}} + O(x^{1-\omega}) + O(x^{-\mu-\nu+\tau}\varphi(x)). \end{aligned} \tag{15}$$

From (14) and (15) the assertion of Theorem 1 follows.

Remark. In Theorems 1 and 2 one may put $0 < \omega \leqslant 1$. However, for $\tau > 0$ and $\omega = 1$, a logarithmic factor will appear in the corresponding remainder term.

The results obtained here cannot be improved. For example, the non-improvability of Theorems 1, 2, and 3 is easily verified by applying them to estimate the distribution function of the zeros of the entire function

$$f(z) = \frac{1}{e^{cz}\Gamma(z)} = \prod_{k=1}^{\infty}\left(1 + \frac{z}{k}\right)e^{-z/k}, \qquad c > 0,$$

where $\Gamma(z)$ is Euler' s gamma function, taking into account the estimate

$$\ln f(z) = z\left(\ln z + c - 1 + \frac{\ln z}{2z}\right) + O(1), \qquad z \in l,\ \tau = 0,\ z \to \infty.$$

In connection with the results obtained here, we note the interesting works ([5,6]).

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