A DIRECT THEOREM ON APPROXIMATION IN THE MEAN OF ANALYTIC FUNCTIONS BY POLYNOMIALS

1969

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. M. KOKILASHVILI

A DIRECT THEOREM ON APPROXIMATION IN THE MEAN OF ANALYTIC FUNCTIONS BY POLYNOMIALS

(Presented by Academician N. I. Muskhelishvili, July 1, 1968)

In the present note we consider the question of the order of polynomial approximation in the mean on the contour of analytic functions of V. I. Smirnov' s class E_p $(p > 1)$. This problem for domains with analytic boundary was solved by J. Walsh and H. Russell (^{1}). S. Ya. Al' per (^{2}) studied the analogous problem under more general assumptions on the smoothness of the boundary of the domain, namely when $\theta(s)$–the angle between the tangent and a fixed direction, expressed as a function of the arc length of the curve–has a modulus of continuity satisfying the condition

$$\int_0^{\varepsilon} \frac{\omega(\delta, \theta)}{\delta}\, d\delta < +\infty, \qquad \varepsilon > 0.$$

In (^{2}) a constructive characterization was obtained of analytic functions of the class E_p $(p > 1)$, whose angular boundary values satisfy an integral Lipschitz condition of order α, $0 < \alpha < 1$. At the same time an analogue of the well-known Jackson inequality was established.

Questions of approximation of analytic functions of the class E_1 were studied in ($^{3-5}$).

In our notes (^{6,7}), exact in order direct and inverse inequalities were established which characterize the order of approximation in the mean by algebraic polynomials of analytic functions of the class E_p $(p > 1)$ in a domain with boundary satisfying condition (1). In the present note we give a direct theorem, exact in order, under general assumptions on the boundary of the domain.

In what follows, $L_p(\Gamma)$ will denote the set of complex-valued measurable functions defined on the curve Γ and satisfying the condition

$$\|f(t)\| = \left\{ \int_{\Gamma} |f(t)|^p |dt| \right\}^{1/p} < +\infty. \tag{1}$$

Definition. A Jordan rectifiable curve Γ will be called a curve of class A if the operator

$$Sf(t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\tau)\,d\tau}{\tau - t},$$

understood in the sense of principal value, is bounded in the space $L_p(\Gamma)$ for $p > 1$.

In the case when Γ is the unit circle, by a well-known theorem of M. Riesz the operator $Sf(t)$ is bounded in L_p $(p > 1)$. B. V. Khvedelidze $(^8)$ generalized this result to the case of Lyapunov curves. Further, in the works $(^{9-12})$ various classes of curves (among them nonsmooth ones) were found along which the operator $Sf(t)$ is bounded in $L_p(\Gamma)$, $p > 1$.

Let G be a simply connected finite domain with rectifiable Jordan boundary Γ. Denote by $w = \psi(z)$ the function conformally mapping–

the exterior of the domain G onto the exterior of the unit disk with the condition $\psi(\infty) = \infty$, $\psi'(\infty) > 0$, and by $z = \varphi(w)$ the inverse function. With the aid of the mapping function, we define the generalized modulus of smoothness in the sense of L_p of the angular boundary values of analytic functions $f(x) \in E_p$ as follows:

$$\omega_k^*(\delta, f)_p = \sup \left\{ \int_0^{2\pi} \left|\Delta_k^h f_0(\theta)\right|^p \, d\theta \right\}^{1/p},$$

where

$$f_0(\theta) = f\left[\varphi(e^{i\theta})\right] \sqrt[p]{\varphi'(e^{i\theta})}, \qquad \Delta_k^h f_0(\theta) = \sum_{\nu=1}^{n} (-1)^{k-\nu} \binom{k}{\nu} f_0(\theta + \nu h).$$

Next, let $\rho_n^{(p)}(f, \Gamma)$ denote the best approximation in the mean on the boundary of the function $f(z) \in E_p$, i.e.

$$\rho_n^{(p)}(f, \Gamma) = \inf \|f(t) - P_k(t)\|$$

over all algebraic polynomials of degree $\leq n$.

Theorem 1. If $f(z)$ is an analytic function of class E_p $(p > 1)$ in the domain G, whose boundary Γ belongs to the class A, then for every natural n there exists a polynomial of degree n such that

$$\|f(t) - P_n(t)\| \leq C_1(p, \Gamma)\omega_k^* \left(\frac{1}{n}, f\right)_p .$$

From Theorem 1 there immediately follows an inequality analogous to the well-known Jackson inequality

$$\rho_n^{(p)}(f, \Gamma) \leq C_1(p, \Gamma)\omega_k^* \left(\frac{1}{n}, f\right)_p . \tag{2}$$

Inequality (2) admits a further refinement in the sense of order.

Theorem 2. For $f(z) \in E_p$, $1 < p < +\infty$, in the domain G with boundary of class A, the inequality

$$\frac{1}{n^k} \left\{ \sum_{\nu=1}^{n} \nu^{\beta k-1} \left[\rho_{\nu-1}^{(p)}(f, \Gamma)\right]^\beta \right\}^{1/\beta} \leq C_2(p, k, \Gamma)\, \omega_k^* \left(\frac{1}{n}, f\right), \tag{3}$$

where $\beta = \max(2, p)$.

In general, estimate (3), for the given class E_p $(p > 1)$, cannot be improved. For $1 < p \leq 2$ this is seen from the following proposition.

Theorem 3. Let $\mathfrak{M}_p(\alpha_\nu)$ denote the class of those functions $f(z) \in E_p$ $(p > 1)$ for which

$$\rho_\nu^{(p)}(f, \Gamma) \sim \alpha_\nu,$$

where $\{\alpha_\nu\}_{\nu=0}^\infty$ is a given monotonically decreasing sequence of numbers tending to zero.

Then

$$\inf_{f \in \mathfrak{M}p(\alpha_\nu)} \omega_k^* \left(\frac{1}{n}, f\right)_p \leq \frac{C_3(p, k, \Gamma)}{n^k} \left\{ \sum_{\nu=1}^{n} \nu^{2k-1} \alpha_{\nu-1}^2 \right\}^{1/2} . \tag{4}$$

As an apparatus for investigating the question of the order of approximation in the mean of analytic functions of class E_p, this time series in generalized Faber polynomials are considered. The mentioned polynomials are defined by the expansion

$$\frac{[\varphi'(w)]^{1-1/p}}{\varphi(w) - z} = \sum_{n=0}^{\infty} \frac{B_n(z)}{w^{n+1}}, \qquad |w| > 1, \quad z \in G. \tag{5}$$

For series in the polynomials $B_n(z)$, assertions are valid analogous to the known theorems of Marcinkiewicz $^{(13)}$ and Littlewood–Paley $^{(14)}$ on multipliers and composition of power series.

Theorem 4. Let $f(z) \in E_p$ $(p > 1)$ in a domain G with boundary of class A, and let a_n be its Faber coefficients, defined by the formula

$$a_n = \frac{1}{2\pi i} \int_{|w|=1} \frac{f[\varphi(w)] \sqrt[p]{\varphi'(w)}}{w^{n+1}} \, dw. \tag{6}$$

If the sequence $\{\lambda_n\}_{n=0}^{\infty}$ of complex numbers satisfies the condition

$$|\lambda_n| \leq M, \qquad \sum_{\nu=2^m}^{2^{m+1}} |\lambda_\nu - \lambda_{\nu+1}| \leq M \qquad (m = 0, 1, 2, ...), \tag{7}$$

then the series

$$\sum_{k=0}^{\infty} \lambda_k a_k B_k(z) \tag{8}$$

converges uniformly in the domain G to an analytic function $F(z) \in E_p$; its angular boundary values almost everywhere coincide with the sum in the mean of the p-th degree of the series (7) on the boundary, and the estimate

$$\|F(t)\| \leq C_4(p, \Gamma) M \|f(t)\| \tag{9}$$

holds.

Theorem 5. For $f(z) \in E_p$ $(p > 1)$ in a domain G with boundary belonging to class A, the estimates

$$C_5(p, \Gamma)\|f(t)\| \leq \left\| \left(\sum_{k=0}^{\infty} |\Delta_k(t)|^2 \right)^{1/2} \right\| \leq C_6(p, \Gamma)\|f(t)\| \tag{10}$$

hold, where

$$\Delta_k(t) = \sum_{\nu=2^{k-1}}^{2^k-1} a_\nu B_\nu(t) \quad \text{for } k \geq 1, \qquad \Delta_0(t) = a_0 B_0(t).$$

Theorems 4 and 5 are used in the proof of Theorem 2. For series in Faber polynomials, when the boundary of the domain satisfies condition (1), analogous theorems were obtained by us earlier $(^{15})$.

Let $\{\lambda_\nu^{(n)}\}$ $(\nu = 0, 1, \dots, n;\ n = 1, 2, \dots;\ \lambda_0^{(n)} = 1,\ \lambda_\nu^{(n)} = 0$ for $\nu > n)$ be a certain triangular matrix of numbers, and let $U_n(f, z, \lambda)$ be the linear operator defined for $f(z) \in E_p$ $(p > 1)$ as follows:

$$U_n(f, z, \lambda) = \sum_{\nu=0}^{n} \lambda_\nu^{(n)} a_\nu B_\nu(z),$$

where a_ν are the Faber coefficients expressed by formula (6).

For each linear operator $U_n(f, z, \lambda)$ consider the quantity

$$R_n(f, \lambda) = \|f(t) - U_n(f, t, \lambda)\|,$$

which characterizes the deviation of the angular boundary values of the function $f(z)$ on Γ from the linear operator $U_n(f, z, \lambda)$.

Theorem 6. Let

$$\mu_\nu^{(n)} = 1 - \lambda_\nu^{(n)} \quad \text{for } 1 \le \nu \le [n^{k/m}],$$

$$\mu_\nu^{(n)} = 0 \quad \text{for } \nu > [n^{k/m}].$$

If

$$\sum_{s=2^\nu}^{2^{\nu+1}} \left| \frac{\mu_{s+1}^{(n)}}{(s+1)^m} - \frac{\mu_s^{(n)}}{s^m} \right| \le \frac{L_1}{n^k} \qquad (\nu = 0, 1, 2, \dots;\ n = 1, 2, \dots), \tag{11}$$

then for $f(z) \in E_p$ $(1 < p < +\infty)$ in a domain with boundary of class A the estimate is valid

$$R_n(f, \lambda) \le C_7(p, \Gamma, \lambda)\, \omega_m^* \left(\frac{1}{n^{[k/m]}}, f \right)_p .$$

From Theorem 6, as corollaries, one obtains estimates of the deviations of Cesàro means of order α, $\alpha > 0$, Zygmund normal means, and many others.

Theorem 7. Let $0 \le \lambda_s^{(n)} = 1$.

If

$$\sum_{\nu=2^s}^{2^{s+1}} \left| \frac{(\nu+1)^k}{\mu_{\nu+1}^{(n)}} - \frac{\nu^k}{\mu_\nu^{(n)}} \right| \le L_2 n^k \qquad (s = 0, 1, 2, \dots;\ n = 1, 2, \dots),$$

then for any $f(z) \in E_p,\ 1 < p < +\infty$, the estimate is valid

$$\frac{1}{n^k}\left\{\sum_{\nu=1}^{n} \nu^{\beta k-1}\,[\rho_\nu^{(n)}(f,\Gamma)]^\beta\right\}^{1/\beta} \le C_8(p,\Gamma)\,\|f(t)-U_n(f,t,\lambda)\|,$$

where $\beta = \max(2,p)$.

Estimates analogous to (10) and (11), for series in Faber polynomials in the case when the boundary satisfies condition (1), were announced in our note (^{6}).

Tbilisi Mathematical Institute named after A. M. Razmadze
Academy of Sciences of the Georgian SSR

Received
21 VI 1968

CITED LITERATURE

[1] J. Walsh, H. Russell, Trans. Am. Math. Soc., **92**, No. 2 (1959).
[2] S. Ya. Al' per, *Studies on Contemporary Problems of the Theory of Functions of a Complex Variable*, Moscow, 1960, pp. 273–286.
[3] M. I. Andrashko, *Problems of Mathematical Physics and the Theory of Functions*, Kiev, No. 1, 1964.
[4] S. Ya. Al' per, Author' s abstract of doctoral dissertation, Rostov-on-Don, 1964.
[5] D. M. Galan, Reports of the Academy of Sciences of the Ukrainian SSR, A, No. 8, 673 (1967).
[6] V. M. Kokilashvili, DAN, **177**, No. 2, 261 (1967).
[7] V. M. Kokilashvili, Communications of the Academy of Sciences of the Georgian SSR, **47**, No. 1, 3 (1967).
[8] V. Khvedelidze, Proceedings of the Tbilisi Mathematical Institute named after A. M. Razmadze, Academy of Sciences of the Georgian SSR, **28** (1956).
[9] A. G. Dzhvarsheishvili, Proceedings of Tbilisi State University, **84** (1961).
[10] E. G. Gordadze, Communications of the Academy of Sciences of the Georgian SSR, **37**, 3 (1965).
[11] I. I. Danilyuk, V. Yu. Shelepov, DAN, **174**, No. 2, 514 (1967).
[12] V. P. Khavin, Vestnik Leningrad University, No. 7, 103 (1967).
[13] J. Marcinkiewicz, Studia Math., **8**, 78 (1939).
[14] J. Littlewood, R. Paley, Proc. London Math. Soc., **42**, 52 (1936).
[15] V. Kokilashvili, Bull. Acad. Polon. Sci., Ser. sci. math., **15**, No. 4 (1967).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.