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## Abstract

## Full Text

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*MATHEMATICS*

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# ON THE RATE OF CONVERGENCE OF DIFFERENTIATED LAGRANGE AND HERMITE INTERPOLATION POLYNOMIALS

*(Presented by Academician M. A. Lavrent'ev on 18 II 1969)*

Let the function  $f(x)$  belong to the class  $W^{(1)}H_\omega$  on the segment  $[-1, 1]$ , i.e., have a first derivative with modulus of continuity  $\omega(\delta, f')$ , not exceeding the given modulus of continuity  $\omega(\delta)$ , and let  $M_1$  be a matrix of nodes with  $(n+1)$ -st row

$$x_k^{(n+1)} = \cos k\pi/n \quad (k = 0, \dots, n). \quad (1)$$

In our paper (2), under these conditions the estimate

$$|f'(x) - L'_n(x, f, M_1)| \leq c_1 \omega(1/n) \ln n, \quad x \in [-1, 1], \quad (2)$$

was established, where the constant  $c_1$  does not depend on the function  $f(x)$ , and  $L_n(x, f, M_1)$  is the Lagrange interpolation polynomial with nodes (1).

In the present paper it is shown that estimate (2) cannot be improved in order, if the entire class of functions  $W^{(1)}H_\omega$  is considered.

We shall assume that the modulus of continuity  $\omega(\delta)$ , as usual (1, p. 108), is defined on the segment  $[0, 2]$ , is continuous and nondecreasing on it, with  $\omega(0) = 0$ , and satisfies the subadditivity condition

$$\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2).$$

Further, putting  $x = \cos \theta$  for  $0 \leq \theta \leq \pi$ , introduce the notation

$$\theta_k^{(n+1)} = \theta_k = k\pi/n, \quad x_k^{(n+1)} = x_k = \cos \theta_k \quad (k = 0, \dots, n),$$

$$\begin{aligned}
 u_k^{(n)} = u_k = (x_k - x_{k+1})/2, \quad y_k^{(n)} = y_k = \\
 = (x_k + x_{k+1})/2 \quad (k = 0, \dots, n-1);
 \end{aligned}
 \tag{3}$$

$$L_n(x, f, M_1) = \sum_0^n f(x_k) l_{k,n}(x),$$

$$l_{k,n}(x) = (-1)^{k+1} \frac{\sin n\theta \sin \theta}{n(x-x_k)} \quad (k = 1, \dots, n-1); \tag{4}$$

$$l_{i,n}(x) = (-1)^{i+1} \frac{\sin n\theta \sin \theta}{2n(x-x_i)} \quad (i = 0; n).$$

Let  $p = [n/2]$  be the integer part of the number  $n/2$ . For every  $n$  define the function  $F_n(x)$  as follows:

$$F_n(x) = \frac{(-1)^{p-1}}{12} \int_0^{x-x_p} \omega(t) dt + d_p, \quad x \in [x_p, y_{p-1}],$$

$$F_n(x) = \frac{(-1)^k}{12} \int_0^{|x-x_{p+1}|} \omega(t) dt + d_{k+1}, \quad x \in [y_{k+1}, y_k] \quad (k = 1, \dots, p-2),$$

$$F_n(x) = -\frac{1}{12} \int_0^{x_1-x} \omega(t) dt, \quad x \in [y_1, x_1].$$

Further, let  $F_n(x) = 0$  for  $x \in [x_1, 1]$ ; let  $F_n(x) = (-1)^{n+1} F_n(-x)$  for  $x \in [-1, x_{p+1}]$ ; and, finally, for  $x \in [x_{p+1}, x_p]$  set  $F_n(x) = H_n(x)$ ,

where  $H_n(x)$  is the Hermite interpolation polynomial (of the third degree), coinciding with the values of the function  $F_n(x)$  and of its derivative at the points  $x_{p+1}$  and  $x_p$ .

**Lemma 1.** For fixed  $n$  there exists a system of numbers  $\{d_k\}$  ( $k = 2, \dots, p$ ) such that the function  $F_n'(x)$  belongs to the class  $W^{(1)}H_\omega$  on the segment  $[-1, 1]$ .

**Proof.** For continuity of the function  $F_n(x)$  at the point  $y_k$ , it is necessary and sufficient that the relations

$$6d_{k+1} = \sum_1^k (-1)^{i-1} \int_0^{u_i} \omega(t) dt \quad (k = 1, \dots, p-1). \tag{5}$$

be fulfilled.

From the equalities (5) and the choice (3) of the points  $y_k$  ( $k = 0, \dots, n - 1$ ) it follows that the derivative  $F'_n(x)$  is also continuous on the segment  $[-1, 1]$ . It is not difficult to see that  $F'_n(x) \in H_\omega$ . Indeed, let, for example,  $x', x'' \in [x_{p+1}, x_p]$  ( $x'' > x'$ ); then from the definition of the function  $F_n(x)$  it follows that

$$|F'_n(x'') - F'_n(x')| = |H'_n(x'') - H'_n(x')|. \quad (6)$$

But  $H_n(x) = d_p$  for  $n = 2p + 1$ , and therefore

$$|F'_n(x'') - F'_n(x')| \leq \omega(x'' - x'). \quad (7)$$

For  $n = 2p$  we shall have

$$H_n(x) = (d_p + d_{p-1})(2x^3/x_{p+1}^3 - 3x^2/x_{p+1}^2) + d_p.$$

Now, by virtue of the estimate ( $n = 2p$ )

$$|d_p + d_{p-1}| < 4 \sin^2 \frac{\pi}{2n} \omega\left(\left|\frac{x_{p+1}}{2}\right|\right)$$

and the relation ((1), p.111)

$$\omega(\delta_2)/\delta_2 \leq 2\omega(\delta_1)/\delta_1 \quad (\delta_2 > \delta_1) \quad (8)$$

from (6) we obtain (7) also for  $n = 2p$ .

If  $x' \in [x_{k+1}, y_k]$ ,  $x'' \in [x_{i+1}, y_i]$  ( $i, k = 1, \dots, p - 1$ ), then for  $i = k$  it follows from the definition of the function  $F_n(x)$  that

$$|F'_n(x'') - F'_n(x')| = \frac{1}{12} |\omega(x'' - x_{k+1}) - \omega(x' - x_{k+1})| \leq \frac{1}{12} \omega(x'' - x')$$

by virtue of the semiadditivity of the function  $\omega(\delta)$ ; for the same  $x'$  and  $x''$  in the case  $i \leq k - 1$  we have

$$|F'_n(x'') - F'_n(x')| = \frac{1}{12} |\omega(x'' - x_{i+1}) - (-1)^{k+i} \omega(x' - x_{k+1})|;$$

but for  $i \leq k - 1$  one has  $x'' - x_{i+1} < x'' - x'$ ,  $x' - x_{k+1} < x'' - x'$ , and then by the nondecreasing property of  $\omega(\delta)$  inequality (7) will hold. The validity of relation (7) for other possible cases of the location of the points  $x'$  and  $x''$  is established similarly.

**Lemma 2.** For  $k = 1, \dots, p - 2$ , for the numbers  $d_{k+1}$  defined by equalities (5), the estimates

$$|d_{k+1}| > \frac{1}{12} u_k \omega(u_k) [1/8 - 2u_k + \sin^4 \pi/2n]. \quad (9)$$

are valid.

These estimates are obtained from (5), taking into account the nondecreasing property of  $\omega(\delta)$  and inequality (8).

**Theorem 1.** For the function  $F_n(x)$  the inequality

$$\max_{x \in [-1, 1]} |F'_n(x) - L'_n(x, F_n, M_1)| \geq c_2 \omega\left(\frac{1}{n}\right) \ln n + O\left(\omega\left(\frac{1}{n}\right) \frac{\ln n}{n}\right), \quad (10)$$

holds, where  $c_2$  and the constant in the estimate of the remainder term do not depend on  $n$ .

**Proof.** Let  $x = x_p$ ; then from (5) we obtain  $F'_n(x) = 0$ . For  $n = 2p$ , from (4) we have

$$L'_n(x, F_n, M_1) = \sum_2^{p-1} [F_n(x_k) l'_{k,n}(x) + F_n(x_{n-k}) l'_{n-k,n}(x)] + F_n(x_p) l'_{p,n}(x),$$

whence, taking into account the definition of the function  $F_n(x)$ , the equality  $l'_{p,n}(x_p) = 0$  ( $n = 2p$ ), and (5), (4), we find

$$|L'_n(x, F_n, M_1)| = \sum_1^{p-2} |d_{k+1}| \left( \frac{1}{x_{k+1} - x} + \frac{1}{x_{k+1} + x} \right).$$

Now, with the aid of (9), we obtain

$$|L'_n(x, F_n, M_1)| \geq \frac{1}{96\pi} \omega\left(\frac{1}{n}\right) \ln n + O\left(\omega\left(\frac{1}{n}\right) \frac{\ln n}{n}\right), \quad (11)$$

where the constant in the estimate of the remainder term does not depend on  $n$ . The case  $n = 2p + 1$  is considered analogously.

Thus, Theorem 1 is proved.

Let, as before,  $f(x)$  belong to the class  $W^{(1)}H_\omega$  on the segment  $[-1, 1]$ . In the paper (2) the estimate

$$|f'(x) - H'_{2n+1}(x, f, M_1)| \leq c_3 \omega\left(\frac{1}{n}\right) \ln n, \quad x \in [-1, 1], \quad (12)$$

was established, where the constant  $c_3$  does not depend on the function  $f(x)$ , and  $H_{2n+1}(x, f, M_1)$  is the Hermite interpolation polynomial with double nodes at the points (1).

It is not difficult to show that the estimate (12) likewise cannot be improved in order in the class of functions  $W^{(1)}H_\omega$ . Put

$$z_k = \cos \frac{2k-1}{4n} \pi \sec \frac{\pi}{4n} \quad (k = 1, \dots, 2n),$$

$$u_k = \frac{z_k - z_{k+1}}{2}, \quad y_k = \frac{z_k + z_{k+1}}{2} \quad (k = 1, \dots, 2n-1); \quad (13)$$

$$H_{2n+1}(x, f, M_1) = \sum_0^n \{f(x_k)A_{k,n}(x) + f'(x_k)B_{k,n}(x)\},$$

$$A_{k,n}(x) = \left[ 1 + \frac{x_k}{\sin^2 \theta_k} (x - x_k) \right] l_{k,n}^2(x); \quad (14)$$

$$B_{k,n}(x) = (x - x_k)l_{k,n}^2(x).$$

**Theorem 2.** On the segment  $[-1, 1]$  there exists a function  $\Phi_n(x)$  of the class  $W^{(1)}H_\omega$  such that the inequality

$$\max_{x \in [-1, 1]} |\Phi'_n(x) - H'_{2n+1}(x, \Phi_n, M_1)| \geq c_4 \omega\left(\frac{1}{n}\right) \ln n + O\left(\omega\left(\frac{1}{n}\right) \frac{1}{n}\right), \quad (15)$$

holds, where  $c_4$  and the constant in the estimate of the remainder term do not depend on  $n$ .

For the proof of (15) it suffices, with the notation (13), to consider the function  $\Phi_n(x)$  defined as follows:

$$\Phi_n(x) = \Phi_n(-x), \quad x \in [-1, 0]; \quad \Phi_n(x) = b_n, \quad x \in [0, z_n];$$

$$\Phi_n(x) = \frac{(-1)^{n-1}}{2} \int_0^{x-z_n} \omega(t) dt + b_n, \quad x \in [z_n, y_{n-1}];$$

$$\Phi_n(x) = \frac{(-1)^k}{2} \int_0^{|x-z_{k+1}|} \omega(t) dt + b_{k+1}, \quad x \in [y_{k+1}, y_k] \quad (k = 1, \dots, n-2).$$

$$\Phi_n(x) = \frac{1}{2} \int_0^{z_1-x} \omega(t) dt, \quad x \in [y_1, 1].$$

Here, as in the proof of Lemma 1, one first establishes the existence of a system of numbers  $\{b_k\}$  ( $k = 2, \dots, n$ ) ensuring the continuity of  $\Phi_n(x)$ , and then the membership of  $\Phi_n(x)$  in the class  $W^{(1)}H_\omega$ . Next, slightly modifying the reasoning in Theorem 1, with the aid of the notation (1), (4), (13), (14), it is not difficult to establish the estimate (15).

Analogous results hold if the interpolation nodes are the zeros of the polynomials

$$(x^2 - 1)P_{n-1}^{(\alpha, \beta)}(x),$$

where  $P_{n-1}^{(\alpha, \beta)}(x)$  are Jacobi polynomials under the conditions  $|\alpha| = |\beta| = 1/2$ , different from those considered above.

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*Note: Figure translations are in progress. See original paper for figures.*

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