ON THE SUPERPOSITION OF ABSOLUTELY CONTINUOUS FUNCTIONS AND ON THE SUPERPOSITION OF FUNCTIONS OF BOUNDED VARIATION

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.51

MATHEMATICS

V. I. BURENKOV

ON THE SUPERPOSITION OF ABSOLUTELY CONTINUOUS FUNCTIONS AND ON THE SUPERPOSITION OF FUNCTIONS OF BOUNDED VARIATION

(Presented by Academician A. N. Kolmogorov on April 2, 1969)

The question of the superposition of absolutely continuous functions was studied in detail in the work of G. M. Fichtenholz ([1]) and in the works of N. K. Bari (see the detailed exposition ([2])), who also investigated the question of the superposition of functions of bounded variation ([3]).

In ([1]) it was established, in particular, that in order that, for every absolutely continuous function $\Phi(x)$, the superposition $\Phi(f(x))$ be absolutely continuous, it is necessary and sufficient that the function $f(x)$ be absolutely continuous and that the solution of the equation $f(x) = y_0$ consist of isolated points and intervals, the total number of which does not exceed a number N independent of y_0. It follows from this that, by improving the differential properties of the function $f(x)$, we cannot, generally speaking, ensure that the superposition $\Phi(f(x))$ becomes absolutely continuous. In particular, one can choose Φ and f so that f is infinitely differentiable, Φ is an absolutely continuous function, but the superposition $\Phi(f(x))$ is not absolutely continuous. An analogous example can also be constructed in the case when Φ is a function of bounded variation.

On the other hand, it is known that the product $\Phi(f(x))f'(x)$ may possess better properties than the factor $\Phi(f(x))$. For example, if $\Phi(x)$ is a summable function and $f(x)$ is absolutely continuous and monotone, then the product $\Phi(f(x))f'(x)$ is a summable function, although $\Phi(f(x))$ may fail to be a summable function (see, for example, ([4]), p. 284). Let us also note that if $\Phi(x)$ is a bounded measurable function and $f(x)$ is absolutely continuous, then $\Phi(f(x))f'(x)$ is a summable function (([5]), p. 302).

In the present paper it is clarified to what extent the properties of the product $\Phi(f(x))f'(x)$ improve in comparison with the factor $\Phi(f(x))$ in the case when Φ is an absolutely continuous function or a function of bounded variation.

In what follows we shall assume that the function $f(x)$ is defined on the interval

$[a, b]$, and $\Phi(x)$ on the interval $[c, d]$, where

$$c = \min_{x\in[a,b]} f(x), \qquad d = \max_{x\in[a,b]} f(x).$$

Theorem 1. Let $\Phi(x)$ be absolutely continuous on $[c, d]$, and let $f(x)$ be differentiable on $[a, b]$ and let the derivative $f'(x)$ be absolutely continuous. Then the function $\Phi(f(x))f'(x)$ is absolutely continuous on $[a, b]$.

It has already been noted above that the factor $\Phi(f(x))$ need not be absolutely continuous. We illustrate this fact by a simple example. Let

$$f_{\alpha,\beta}(x) = x^\alpha \left(\sin\frac{1}{x^\beta} + 2\right)$$

for $0 < x \leq 1$, $f_{\alpha,\beta}(0) = 0$, where $\alpha > 0$, $\beta > 0$, and $\Phi(x) = x^\gamma$, $0 \leq x \leq 3$, $0 < \gamma < 1/2$. It is not hard to verify that if one chooses $\beta > \gamma/(1 - 2\gamma)$ and $1 + 2\beta < \alpha \leq \beta/\gamma$, then the derivative $f'(x)$ is absolutely continuous; consequently, by Theorem 1 the product $\Phi(f(x))f'(x)$ is absolutely continuous, whereas the factor $\Phi(f(x))$ is not an absolutely continuous function.

Below we give a less trivial example, which shows to what extent "bad"differential properties may be possessed by the factor $\Phi(f(x))$ in Theorem 1. Let us note that, by virtue of Banach' s theorem ([6]), p. 169, the factor $\Phi(f(x))$ in Theorem 1 has a derivative on a set $E \subset [a, b]$ such that $\operatorname{mes} E \cap (\alpha, \beta) > 0$ for every interval $(\alpha, \beta) \subset [a, b]$. On the other hand, for any perfect nowhere dense set $P \subset [a, b]$ (whose measure may be arbitrarily close to $b - a$) one can specify Φ and f, satisfying the condition of Theorem 1, such that the superposition $\Phi(f(x))$ is not differentiable on the set P (in particular, if $\operatorname{mes} P > 0$, then the function $\Phi(f(x))$ is certainly not absolutely continuous).

The functions Φ and f are constructed as follows. Let $\varphi(x) \in C^\infty[0, 1]$, $\varphi^{(n)}(0) = \varphi^{(n)}(1) = 0$, $n = 0, 1, ...$, $0 < \varphi(x) \leq 1$ for $x \in (0, 1)$, $\varphi(1/2) = 1$.

Let, further,

$$(a, b) - P = \sum_{k=1}^{\infty} (a_k, b_k), \quad \delta_k = b_k - a_k, \quad \text{where } \delta_1 \geq \delta_2 \geq \cdots.$$

Denote by r_n the length of the largest of the intervals of the set

$$[a, b] - \sum_{k=1}^{n} (a_k, b_k).$$

It is not difficult to prove that $r_n \downarrow 0$ as $n \to \infty$. Put $f(x) = 0$ on P and $f(x) = \delta_k^2 \varphi[(x - a_k)/\delta_k]$ on (a_k, b_k). As $\Phi(x)$, $0 \leq x \leq 1$, take a monotonically increasing function, differentiable for all $0 < x \leq 1$, for which $\Phi(0) = 0$, $\Phi(\delta_n^2) = \delta_n + r_n$. In the case when f is an absolutely continuous function, an example of analogous content was constructed by N. K. Bari ([2]), p. 212.

A theorem analogous to Theorem 1 is also valid in the case when Φ is a function of bounded variation.

Theorem 2. Let $\Phi(x)$ have bounded variation on $[c,d]$, $f(x)$ be differentiable on $[a,b]$, and the derivative $f'(x)$ have bounded variation. Then the function $\Phi(f(x))f'(x)$ has bounded variation.

It should be noted that if $\Phi(x)$ is a summable function, then the theorem analogous to Theorems 1 and 2 is false for it (although, as was noted at the beginning of the article, an improvement of the properties of the product $\Phi(f(x))f'(x)$ in comparison with the factor $\Phi(f(x))$ is observed). This can be verified by again considering the functions $f(x) = f_{\alpha,\beta}(x)$ and $\Phi(x) = x^{\gamma}$. If $-1 < \gamma < 0$ and $1 < \beta < \alpha \leq \beta/(1+\gamma)$, then $\Phi(x)$ is summable, the derivative $f'(x)$ exists everywhere on $[0,1]$ and is summable, but the product $\Phi(f(x))f'(x)$ is not summable.

The proof of Theorem 2 consists of two parts. First it is proved that

$$\operatorname*{Var}_{[a,b]} \Phi(f(x))f'(x) \leq M \operatorname*{Var}_{[a,b]} f'(x) + \sum_i \operatorname*{Var}_{[a_i,b_i]} \Phi(f(x))f'(x), \tag{1}$$

where

$$M = \sup_{x\in[a,b]} |\Phi(f(x))|,$$

and the intervals $[a_i,b_i]$ are the closures of the component intervals (a_i,b_i) of the open set

$$S = \{x:\ x \in (a,b),\ f'(x) \neq 0\}.$$

Next, the second term on the right-hand side of inequality (1) is estimated. As a result we arrive at the estimate

$$\operatorname*{Var}_{[a,b]} \Phi(f(x))f'(x) \leq \begin{cases} M \operatorname{Var}_{[a,b]} f'(x) + m \operatorname{Var}_{[c,d]} \Phi(x), & \text{if } f'(x) \neq 0 \text{ on } [a,b], \\ \Big[2M + \operatorname{Var}_{[c,d]} \Phi(x)\Big] \operatorname{Var}_{[a,b]} f'(x), & \text{otherwise} \end{cases} \tag{2}$$

$$\left(m = \sup_{x\in[a,b]} |f'(x)| \right),$$

from which Theorem 2 follows.

For the proof of Theorem 1 we use the Banach–Zarecki theorem (see ([7]) or ([4]), p. 269), according to which a continuous function $\varphi(x)$, $x \in [a,b]$, of bounded variation is absolutely continuous if and only if

and only if it has property N on $[a,b]$, i.e., for any $M \subset [a,b]$, $\operatorname{mes} M = 0$ implies $\operatorname{mes} \varphi(M) = 0$. Theorem 1 follows from this theorem, Theorem 2, and the following assertion: if f and g are continuous and have property N, then the product fg also has property N.

Theorem 1 can be used in certain questions on equations in partial derivatives in the following situation (we give the simplest example). In a number of cases it is necessary to integrate by parts the expression

$$\int_a^b \frac{d^2u}{dx^2}\, v\, dx,$$

where the function $u(x)$ is twice continuously differentiable, and $v(x)$ is chosen in a special way. Sometimes it is convenient to put $v(x) = \Phi(u(x))$, where Φ is an absolutely continuous function (for example, $\Phi(t) = (|t|^p)'$, $1 < p < \infty$; here the case $1 < p < 2$ is of interest). The superposition $\Phi(u(x))$ need not be absolutely continuous. In this case we shall use the fact that the formula for integration by parts

$$\int_a^b f' g\, dx = fg\big|_a^b - \int_a^b fg'\, dx$$

is valid under the following assumptions: 1) f' and g' exist almost everywhere, 2) one of the products $f'g$ or fg' is summable, and 3) the product fg is absolutely continuous. In the case under consideration $f = du/dx$, $g = \Phi(u)$. By Theorem 1 the product $fg = \Phi(u)\, du/dx$ is absolutely continuous. Therefore

$$\int_a^b \frac{d^2u}{dx^2}\, \Phi(u)\, dx = \Phi(u)\frac{du}{dx}\bigg|_a^b - \int_a^b \Phi'(u)\left(\frac{du}{dx}\right)^2 dx$$

(if it is known that almost everywhere $(\Phi(u))' = \Phi'(u)u'$).

Moscow Institute of Radio Engineering,
Electronics and Automation

Received
26 III 1969

REFERENCES

1. G. M. Fichtenholz, *Mat. sbornik*, 31, 286 (1924).

2. H. K. Bari, *Math. Ann.*, 103, 185 (1930); 103, 598 (1930).

3. H. K. Bari, *Mat. sbornik*, 40, 326 (1933).

4. I. P. Natanson, *Theory of Functions of a Real Variable*, Moscow, 1957.

5. III. Vallee-Poussin, *Cours d' analyse infinitésimale*, 1, Moscow–Leningrad, 1933.

6. S. Banach, *Fund. Math.*, 8, 166 (1926).

7. M. A. Zaretskii, *Dokl. Akad. Nauk* (A), 88 (1925).

Note: Figure translations are in progress. See original paper for figures.