NECESSARY AND SUFFICIENT CONDITIONS FOR THE REGULARITY OF A BOUNDARY POINT FOR THE DIRICHLET PROBLEM FOR THE HEAT EQUATION

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE REGULARITY OF A BOUNDARY POINT FOR THE DIRICHLET PROBLEM FOR THE HEAT EQUATION

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In this note necessary and sufficient conditions will be obtained for the regularity of a boundary point for the Dirichlet problem for the equation

$$Lu \equiv \sum_{i,k=1}^{n} a_{ik}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial u}{\partial x_i} + c(t,x)u - \frac{\partial u}{\partial t} = 0 \qquad (1)$$

in an arbitrary domain D of the space t, x, $x = (x_1, \dots, x_n)$.

With respect to the coefficients of the equation it is assumed that:

$$C_1|\xi|^2 \leqslant \sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k \leqslant C_2|\xi|^2, \qquad C_1, C_2 > 0, \qquad (2)$$

all the coefficients are bounded and satisfy a Hölder condition.

In the case when the domain D is a cylinder, for the heat equation this was done by A. N. Tikhonov ([1]). For the one-dimensional heat equation and for a domain bounded by two straight lines parallel to the t-axis and by the curves $x = \varphi_1(t)$ and $x = \varphi_2(t)$, sufficient conditions and necessary conditions, very close to one another but not completely coinciding, were obtained by I. G. Petrovskii ([2]). In the multidimensional case, the question of the regularity of a boundary point when at this point the boundary is tangent to the characteristics was not considered for a long time. In recent years interest in these questions has revived. On the one hand, the behavior of the solution of the Dirichlet problem for a parabolic equation began to be studied at the very highest and the very lowest, with respect to the t-axis, points of the domain, when in neighborhoods of these points the boundary behaves like a paraboloid ([3,4]). On the other hand, the question of the regularity of a boundary point for a parabolic equation began to be considered as a special case of the more general problem of degenerate elliptic-parabolic equations ([4,6,7]), etc. In these works the boundary was assumed sufficiently smooth and, in particular, the question of the relation between the smoothness of the solution at such a point and the character of the boundary was investigated. In the present note an arbitrary boundary is considered, but only the question of the continuity of the solution at a boundary point is studied.

1°. Some definitions and notation. A domain D^0, whose boundary Γ^0 is a polyhedron, will be called **simple**.

For a simple domain consider the Dirichlet problem

$$Lu = 0 \text{ in } D^0, \qquad u|_{\Gamma^0} = \varphi, \qquad \varphi \text{ continuous on } \Gamma^0. \tag{3}$$

By a solution of such a problem we shall mean a function satisfying equation (1) in D^0, continuous on Γ^0 except for the $(n+1-k)$-dimensional faces ($k = 1, \dots, n+1$) orthogonal to the t-axis (if $k = n+1$, i.e. the face is a vertex, then we shall include here those vertices which have a supporting hyperplane orthogonal to the t-axis), which cannot be touched from inside the domain from above, bounded and assuming the boundary val-

φ everywhere, except on the indicated faces. Such a solution exists and is unique.

Let an arbitrary bounded domain D with boundary Γ be given, and consider for it the Dirichlet problem:

$$Lu = 0, \qquad u|_{\Gamma} = \varphi, \qquad \varphi \text{ is continuous on } \Gamma. \tag{4}$$

We construct a generalized solution of this problem by Wiener' s method: approximate D from inside by simple domains D^0_m. Extend φ continuously to $\overline{D}$. Consider the sequence of problems

$$Lu_m = 0, \qquad u_m|_{\Gamma^0_m} = \varphi$$

(Γ^0_m is the boundary of D^0_m). The sequence $\{u_m\}$ converges at each point of D to a certain function $u(t,x)$. The function $u(t,x)$ is a solution of equation (1), is bounded, and depends neither on the method of extending the function φ nor on the choice of the sequence $\{D^0_m\}$. This function is called the **generalized solution of the Dirichlet problem** (4).

A point (t,x) of the boundary Γ of the domain D is called **regular** for the Dirichlet problem for equation (1) if, for every continuous φ prescribed on the boundary, the generalized solution of problem (4) assumes at this point its boundary value.

Let E be a B-set in the $(n+1)$-dimensional space (t,x). Let the point (t^0,x^0) be such that E lies entirely in the half-space $t<t^0$. Consider all possible measures μ, defined on E, such that

$$\int_E G(t,x;\tau,\xi)\,d\mu(\tau,\xi)\le 1 \qquad \text{for } (t,x)\notin E, \tag{5}$$

where

$$G(t,x;\tau,\xi)=\begin{cases}\dfrac{1}{(t-\tau)^{n/2}}\exp\left[-\dfrac{|x-\xi|^2}{4(t-\tau)}\right], & \text{for } t>\tau,\\ 0, & \text{for } t\le\tau.\end{cases}$$

We shall call the number

$$\gamma(E,t^0,x^0)=\sup\int_E G(t^0,x^0;\tau,\xi)\,d\mu(\tau,\xi)$$

the **potential of the set E at the point** (t^0,x^0), where the least upper bound is taken over all measures satisfying condition (5).

Define two sequences of numbers $\{\rho_m\}$ and $\{\tau_m\}$. Define the sequence $\{\rho_m\}$ recursively:

$$\rho_0=1/2, \qquad \rho_{m+1}=\rho_m/\sqrt{|\ln\rho_m|}.$$

Define the numbers τ_m through ρ_m from the condition

$$4n\tau_m|\ln\tau_m|=\rho_{m-2}^2.$$

Let the point (t^0,x^0) belong to the boundary of the domain D. Denote by H the complement of the domain D, and let H_m be the intersection of H with the strip

$$t^0-\tau_m\le t<t^0-\tau_{m+1}.$$

Denote by γ_m the potential of the set H_m with respect to the point (t^0,x^0).

2°. **Theorem.** *In order that the point $(t^0,x^0)\in\Gamma$ be a regular boundary point for problem (4), it is necessary and sufficient that*

$$\sum_{m=1}^{\infty}\gamma_m=\infty. \tag{6}$$

We shall give here the main points of the proof for the case when (1) is the equation

$$\Delta u - \partial u/\partial t = 0. \tag{7}$$

For convenience we shall assume that (t^0, x^0) coincides with the origin of coordinates.

Introduce the notation: $n' = \max(n, 3)$; Π_m is the cylinder $|x| < \rho_{m-n}$, $-\tau_m \le t < 0$; Π_m^1 is the cylinder $|x| < \rho_m$, $-\tau_m \le t < -\tau_{m+1}$; Π_m^2 is the cylinder $|x| < \rho_{m+n}$, $-\tau_{m+n} \le t < 0$; S_m^1 is the lateral surface of the cylinder Π_m^1.

Lemma 1. Put $E_m = H_m \setminus \Pi_m^1$. Then

$$\sum_{m=1}^{\infty} \gamma(E_m, 0, 0) < \infty. \tag{8}$$

The proof of this lemma is obtained by a simple calculation.

Let D^0 be a domain contained in D, intersecting the lateral surface or the lower base of the cylinder Π_m. Let $D_m^0 = D^0 \cap \Pi_m$, $D_m^2 = D^0 \cap \Pi_m^2$, $H_m^1 = H_m \cap \Pi_m^1$. Let μ be a measure defined on H_m^1 and such that

$$U_m(t, x) = \int_{H_m^1} G(t, x; \tau, \xi)\, d\mu(\tau, \xi) \le 1 \quad \text{for } (t, x) \in H_m^1. \tag{9}$$

Let $U_m(0, 0) = \gamma_m^0$.

Lemma 2. Let in D_m there be defined a solution u of equation (7), positive in D_m^0, continuous in D_m^0 and vanishing on that part Γ_m^0 of the boundary D_m^0 which is situated strictly inside Π_m. Then

$$\sup_{(t,x)\in D_m^0} u(t, x) > (1 + \eta\gamma_m^0) \sup_{(t,x)\in D_m^2} u(t, x) \tag{10}$$

for $m > m_0$, where $\eta > 0$ and m_0 depend on n.

We give the main points of the proof of this lemma. It is shown that, for sufficiently large m,

$$\sup_{(t,x)\in S_m^1} U_m(t, x) \le \frac{1}{2} \inf_{(t,x)\in\Pi_m^2} U_m(t, x), \tag{11}$$

$$\inf_{(t,x)\in\Pi_m^2} U_m(t, x) \ge \frac{1}{2}\gamma_m^0. \tag{12}$$

Put $\sup_{(t,x)\in\Pi_m} u(t,x) = M$ and construct the auxiliary function

$$v = M\left[1 - U_m + \frac{1}{2}\inf_{\Pi_m^2} U_m\right].$$

We then have $v|_{S_m} \geq u|_{S_m}$ (by (9) and (11)); $v|_{\Gamma_m^0} \geq u|_{\Gamma_m^0}$ (since $v|_{\Gamma_m^0} > 0$ and $u|_{\Gamma_m^0} = 0$); and, finally, for $t = -\tau_m$ outside H_m^1 we also have $u > v$, since there $U_m = 0$ and therefore $v > M$. Hence, by the maximum principle, $v > u$ in D_m^0, and therefore

$$\sup_{\Pi_m^2} u \leq M\left(1 - \frac{1}{2}\inf_{\Pi_m^2} U_m\right),$$

which, taking into account (11) and (12), gives the required inequality (9).

To prove the regularity of the point $(0,0)$, it is enough to show the following: let ε and K be two positive numbers and let m' be a natural number. There exists $\delta > 0$ such that, whatever the simple domain D^0 intersecting the δ-neighborhood of the origin and the hyperplane $t = -\tau_{m'}$, and whatever the solution $u(t,x)$ of equation (7), bounded above by the constant K and vanishing on that part of the boundary of D^0 which is situated in the open strip $-\tau_{m'} < t < 0$, inside the intersection of D^0 with the δ-neighborhood of the origin, the inequality will hold—

inequality $u(t,x) < \varepsilon$. From the divergence of the series (6) and Lemma 1 it follows that at least one of the series $\sum_{m=1}^{\infty} \gamma_m^0 + (2n'+l)m, \quad l = 0, 1, \dots, 2n-1,$ diverges. Let, for definiteness, this be the series $\sum_{m=1}^{\infty} \gamma_{m'+2mn'}^0$. Let m'' be so large that

$$\prod_{m=1}^{m''} (1 + \eta\gamma_{m'+2n'm}^0) > \frac{K}{\varepsilon}. \tag{13}$$

If now we assume that the intersection D^0 with $\Pi_{m'+2n'm''}$ is nonempty and that

$$\sup_{(t,x)\in D_{m'+2n'm''}^0} u(t,x) > \varepsilon, \tag{14}$$

then we arrive at a contradiction: successively applying Lemma 2 to pairs of cylinders $\Pi_{m'+2n'm}$ and $\Pi_{m'+2n'(m-1)}$, we obtain

$$\sup_{\Pi_{m'+2n'(m-1)}} u > (1 + \eta\gamma_{m'+2n'm}^0) \sup_{\Pi_{m'+2n'm}} u,$$

and therefore from (13) and (14) it follows that $\sup u > K$.

3°. We give the scheme of the proof of the necessity of condition (6).

Choose m_1 so large that

$$\sum_{m_1}^{\infty} \gamma_m < \frac{1}{4}.$$

Consider the intersection D_m of the domain D with the strip $-\tau_m < t < -\tau_{m+1}$. It can be shown that in D_m one can inscribe a simple domain D'_m such that, if by H'_m we denote the complement of D'_m in this strip, then the difference $\gamma_m - \gamma(H'_m, 0, 0)$ will be arbitrarily small. Consider those n-dimensional faces of the boundary of D'_m which are situated on the hyperplane $t = -\tau_{m+1}$. Denote by D''_m that part of D'_m which consists of points each of which can be connected with one of the indicated faces by means of a polygonal line lying in D'_m and projecting one-to-one onto the t-axis. Let Γ''_m be the boundary of D''_m. Let K''_m be the aggregate of the n-dimensional faces of Γ''_m lying in the hyperplane $t = -\tau_m$. Remove from Γ''_m these faces, and also the faces orthogonal to the t-axis which cannot be touched from D''_m from above, and denote the remaining part of Γ''_m by S''_m. In D''_m consider the solution v_m of the following first boundary-value problem

$$v_m\big|_{S''_m} = 1, \qquad v_m\big|_{K''_m} = 0.$$

The function v_m can be represented by means of a simple-layer potential with density concentrated on S''_m. Moreover $v_m(0,0) \leqslant \gamma(H'_m, 0, 0)$, and we can obtain

$$\sum_{m=m_1}^{\infty} v_m(0,0) < \frac{1}{2}.$$

After this, with the aid of the maximum principle it is shown that if the boundary function φ of the original problem satisfies the condition $\varphi(t,x) = 0$ for $t < -\tau_{m_1}$, $\varphi(t,x) \leqslant 1$ for $-\tau_{m_1} \leqslant t \leqslant 0$, then

$$\lim_{(t,x)\to(0,0)} u(t,x) < \frac{1}{2}.$$

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Note: Figure translations are in progress. See original paper for figures.