

# APPROXIMATE SOLUTION OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE DERIVATIVES

MATHEMATICS

1969

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**Abstract**

**Full Text**

UDC 517.917

**MATHEMATICS**

**M. A. BELYAEVA**

**APPROXIMATE SOLUTION OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE DERIVATIVES**

*(Presented by Academician L. S. Pontryagin on 24 VII 1969)*

A system of equations in vector form is considered:

$$\varepsilon dx/dt = f(x, y), \quad dy/dt = g(x, y) \tag{1}$$

$$(x = (x^1, \dots, x^k); \quad y = (y^1, \dots, y^l)).$$

The functions  $f(x, y) = \{f^1(x, y), \dots, f^k(x, y)\}$  and  $g(x, y) = \{g^1(x, y), \dots, g^l(x, y)\}$  are defined and several times differentiable with respect to their arguments in some domain of the space of variables  $(x, y)$ ;  $\varepsilon$  is a small positive parameter.

In the present article we consider cases in which the system of fast motions

$$dx/d\tau = f(x, y) \quad (\tau = t/\varepsilon) \tag{2}$$

(here  $y$  is a constant vector) has either a degenerate equilibrium position or a degenerate limit cycle.

Suppose first that, for some  $y = y_0$ , system (2) has an equilibrium position  $x = x_0$ , and that the matrix  $A = \|\partial f^i/\partial x^j\|$  at the point  $(x_0, y_0)$  has a zero eigenvalue of multiplicity one, while all the remaining eigenvalues have negative real parts.

Introduce the eigenvectors  $m = (m^1, \dots, m^k)$  and  $n = (n_1, \dots, n_k)$  of the matrix  $A$  and of the transposed matrix  $A'$ , corresponding to the zero eigenvalue and normalized so that  $m^\alpha n_\alpha = 1$ . We require the fulfillment of the additional conditions:

$$p = n_\delta f_{x^\alpha x^\beta}^\delta(x_0, y_0) m^\alpha m^\beta \neq 0; \quad n_\delta f_{y^\mu}^\delta(x_0, y_0) = 0 \quad (\mu = 1, \dots, l).$$

If, say,  $m^1 \neq 0$ ,  $n_1 \neq 0$ , then the equations  $f^2(x, y) = \dots = f^k(x, y) = \det \|\partial f^i / \partial x^j\| = 0$  determine  $x^1, \dots, x^k$  as functions of  $y$ ; substituting these functions into  $f^1(x, y)$ , we obtain a certain function  $H(y)$  with the following properties:  $H(y_0) = 0$ ,  $\partial H(y_0) / \partial y^i = 0$  ( $i = 1, \dots, l$ ).

We impose on the function  $H(y)$  such conditions under which  $H(y) = 0$  determines, in some neighborhood of the point  $y_0$ , a smooth surface  $\Pi$ , say a surface of the form  $y^1 = h(y^2, \dots, y^l)$ . From the form of the function  $H(y)$  it follows that it preserves a constant sign at points not belonging to the surface  $\Pi$ . If the signs of the functions  $H(y)$  and  $p$  are opposite, then system (2) has one degenerate equilibrium position for  $y$  lying on the surface  $\Pi$ , and two nondegenerate equilibrium positions  $x = \psi_1(y)$  and  $x = \psi_2(y)$  for  $y$  not lying on  $\Pi$ ; moreover, if  $y$  lies on one side of  $\Pi$  (for example,  $y^1 < h(y^2, \dots, y^l)$ ), then  $\psi_1(y)$  is a stable equilibrium position, and if  $y$  lies on the other side of  $\Pi$  ( $y^1 > h(y^2, \dots, y^l)$ ), then  $\psi_2(y)$  is a stable equilibrium position.

We shall assume that the following additional conditions are satisfied:

1. The expressions  $n_\delta \psi_i^\gamma g^\alpha \Big|_{\substack{x=x_0 \\ y=y_0}} \neq 0$  ( $i = 1, 2$ ), and if they have the same sign, then it coincides with the sign of  $p$ .
2. The vector  $g(x_0, y_0)$  is not tangent to the surface  $\Pi$ , for example,

$$g^1 - g^\alpha \partial h / \partial y^\alpha \Big|_{\substack{x=x_0 \\ y=y_0}} > 0 \quad (\alpha = 2, \dots, l). \quad (3)$$

Let the point  $y_1$  be sufficiently close to  $y_0$  and satisfy the inequality  $y_1^1 < h(y_1^2, \dots, y_1^l)$ . Denote by  $Y_1(t)$  the solution of the system  $dy/dt = g(\psi_1(y), y)$  with initial condition  $Y_1(t_1) = y_1$ . It follows from condition (3) that the solution  $Y_1(t)$  at some time  $t = t_2$  falls on the surface  $\Pi$ .

Denote by  $Y_2(t)$  the solution of the system  $dy/dt = g(\psi_2(y), y)$  with initial condition  $Y_2(t_2) = Y_1(t_2)$ , defined on some interval  $[t_2, t_3]$ .

**Theorem 1.** *If the solution  $x(t, \varepsilon), y(t, \varepsilon)$  of system (1) satisfies the condition  $|x(t_1, \varepsilon) - \psi_1(y_1)| = O(\varepsilon)$ ,  $|y(t_1, \varepsilon) - y_1| = O(\varepsilon)$ , then on the interval  $[t_1, t_3]$   $x(t, \varepsilon) \rightarrow X(t)$ ,  $y(t, \varepsilon) \rightarrow Y(t)$  as  $\varepsilon \rightarrow 0$ , where*

$$X(t) = \begin{cases} \psi_1[Y_1(t)], & t_1 \leq t \leq t_2, \\ \psi_2[Y_2(t)], & t_2 < t \leq t_3; \end{cases}$$

$$Y(t) = \begin{cases} Y_1(t), & t_1 \leq t \leq t_2, \\ Y_2(t), & t_2 < t \leq t_3. \end{cases}$$

Let us now consider the case when system (2), for some  $y = y_0$ , has a degenerate limit cycle  $x = \varphi_0(\tau)$  of period  $T_0$ . For system (2) we construct an analogue of the successor function, which here will be a mapping of the  $(k-1)$ -dimensional

space of variables  $u^1, \dots, u^{k-1}$  into itself:  $v = \chi(u, y)$ . We shall assume that the matrix

$$M = \|\partial\chi^i(u_0; y_0)/\partial u^j\| \quad (i, j = 1, \dots, k-1)$$

( $u_0$  is the fixed point of the mapping corresponding to the cycle  $\varphi_0(\tau)$ ) has an eigenvalue equal to unity, of multiplicity one, while all the remaining eigenvalues are less than unity in modulus. Introduce the eigenvectors  $m = (m^1, \dots, m^{k-1})$  and  $n = (n_1, \dots, n_{k-1})$  of the matrix  $M$  and of the transposed matrix  $M'$ , corresponding to the unit eigenvalue and normalized so that  $m^\alpha n_\alpha = 1$ . On the function  $\chi(u, y)$  impose the nondegeneracy conditions:

$$p = n_\delta \chi_{u^\alpha u^\beta}^\delta(u_0, y_0) m^\alpha m^\beta \neq 0,$$

$$q = \{q_1, \dots, q_l\} \neq 0, \quad \text{where} \quad q_\mu = n_\delta \chi_{y^\mu}^\delta(u_0, y_0).$$

It follows from these conditions that the system

$$\chi(u, y) = u, \quad \det \|\partial\chi(u, y)/\partial y - E\| = 0$$

defines in the space  $y$ , in some neighborhood of the point  $y_0$ , a smooth surface  $\Pi$  passing through the point  $y_0$  and possessing the following property: if  $y$  lies on one side of  $\Pi$ , then system (2) has two limit cycles, one of which  $x = \varphi(\tau, y)$  is nondegenerate stable with period  $T(y)$ ; if  $y$  belongs to  $\Pi$ , then system (2) has one degenerate limit cycle; and if  $y$  lies on the other side of the surface  $\Pi$ , then system (2) has no limit cycles.

Let us introduce for consideration the averaged system

$$d\bar{y}/dt = \bar{g}(\bar{y}), \tag{4}$$

where

$$\bar{g}(\bar{y}) = \frac{1}{T(\bar{y})} \int_0^{T(\bar{y})} g(\varphi(\theta, \bar{y}), \bar{y}) d\theta; \quad \bar{g}(y_0) = \frac{1}{T_0} \int_0^{T_0} g(\varphi_0(\theta), y_0) d\theta.$$

We shall assume that the vector  $\bar{g}(y_0)$  is not tangent to the surface  $\Pi$  and is directed toward that side of the space  $y$  where those values of  $y$  are located for which system (2) has no limit cycles.

Let the point  $y_1$  be sufficiently close to  $y_0$ , and let system (2) for  $y = y_1$  have two nondegenerate limit cycles. Denote by  $\bar{y}(t)$  the solution of system (4) with the initial condition  $\bar{y}(t_1) = y_1$ . From the condition imposed on the vector  $\bar{g}(y_0)$  it follows that at some moment of time  $t = t_2$  the solution  $\bar{y}(t)$  reaches the surface  $\Pi$ . Without loss of generality, we may assume that  $\bar{y}(t_2) = y_0$ .

**Theorem 2.** *If the solution  $x(t, \varepsilon), y(t, \varepsilon)$  of system (1) satisfies the condition*

$$|x(t_1, \varepsilon) - \varphi(t_1/\varepsilon, y_1)| = O(\varepsilon), \quad |y(t_1, \varepsilon) - y_1| = O(\varepsilon),$$

then there exists a function  $\theta(t, \varepsilon)$ , smoothly depending on  $t$ , such that as  $\varepsilon \rightarrow 0$

$$|\varepsilon d\theta/dt - 1| \rightarrow 0, \quad |y(t, \varepsilon) - \bar{y}(t)| \rightarrow 0,$$

$$|x(t, \varepsilon) - \varphi(\theta(t, \varepsilon), \bar{y}(t))| \rightarrow 0, \quad t_1 \leq t \leq t_2,$$

and such a  $t_3(\varepsilon) > t_2$ , where  $t_3(\varepsilon) \rightarrow t_2$  as  $\varepsilon \rightarrow 0$ , that the point  $x(t_3(\varepsilon), \varepsilon), y(t_3(\varepsilon), \varepsilon)$  lies outside a small, but  $\varepsilon$ -independent, neighborhood of the closed curve  $x = \varphi_0(\tau), y = y_0$ , and moreover  $|y(t_3(\varepsilon), \varepsilon) - y_0| \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Finally, consider the case in which the function  $\chi(u, y)$  satisfies the conditions  $p \neq 0, q = 0$ . One may introduce a function  $H(y)$ , analogous to the function  $H(y)$  considered in theorem (1). The equation  $H(y) = 0$  defines the surface  $\Pi$ , whose equation can be written in the form  $y^1 = h(y^2, \dots, y^l)$ . We shall assume that the signs of the function  $H(y)$  and of  $p$  are opposite, so that for  $y$  lying on the surface  $\Pi$ , the mapping  $v = \chi(u, y)$  has one fixed point, while system (2) has one degenerate limit cycle  $x = \varphi_0(\tau, y)$ ; whereas if  $y$  does not lie on  $\Pi$ , the mapping  $v = \chi(u, y)$  has two fixed points  $u = \psi_1(y)$  and  $u = \psi_2(y)$ , and system (2) has two limit cycles. Denote the limit cycles of system (2) corresponding to these fixed points by  $x = \varphi_1(\tau, y)$  (stable if  $y^1 < h(y^2, \dots, y^l)$ ) and  $x = \varphi_2(\tau, y)$  (stable if  $y^1 > h(y^2, \dots, y^l)$ ). Introduce into consideration the averaged system

$$\frac{d\bar{y}}{dt} = \bar{g}_i(\bar{y}) \quad (i = 1, 2), \quad \text{where} \quad g_i(\bar{y}) = \frac{1}{T_i(\bar{y})} \int_0^{T_i(\bar{y})} g(\varphi_i(\tau, \bar{y}), \bar{y}) d\tau;$$

$i = 1$ , if  $\bar{y}^1 < h(\bar{y}^2, \dots, \bar{y}^l)$ , and  $i = 2$ , if  $\bar{y}^1 > h(\bar{y}^2, \dots, \bar{y}^l)$ .  $T_i(\bar{y})$  is the period of the cycle  $\varphi_i(\tau, \bar{y})$ ;  $i = 0$ , if  $\bar{y} \in \Pi$ .

We shall assume that the expressions

$$n_\delta \psi_{iy^\alpha}^\delta \bar{g}^\alpha \Big|_{y=y_0} \neq 0 \quad (i = 1, 2),$$

and if they have the same sign, then it coincides with the sign of  $p$ , and that the vector  $\bar{g}(y_0)$  is not tangent to the surface  $\Pi$ , for example,

$$\bar{g}^1 - \bar{g}^\alpha \partial h / \partial y^\alpha \Big|_{y=y_0} > 0 \quad (\alpha = 2, \dots, l). \quad (5)$$

Let the point  $y_1$  be sufficiently close to the point  $y_0$  and satisfy the inequality  $y_1^1 < h(y_1^2, \dots, y_1^l)$ . Denote by  $Y_1(t)$  the solution of the averaged system for  $i = 1$  with the initial condition  $Y_1(t_1) = y_1$ . From condition (5) it follows that the solution  $Y_1(t)$ , at some moment of time  $t = t_2$ , reaches the surface  $\Pi$ . Denote by  $Y_2(t)$  the solution of the averaged system for  $i = 2$  with the initial condition  $Y_2(t_2) = Y_1(t_2)$ , defined on some interval  $[t_2; t_3]$ .

**Theorem 3.** *If the solution  $x(t, \varepsilon), y(t, \varepsilon)$  of system (1) satisfies the conditions*

$$|x(t_1, \varepsilon) - \varphi_1(t_1/\varepsilon, y_1)| = O(\varepsilon), \quad |y(t_1, \varepsilon) - y_1| = O(\varepsilon),$$

then there exists a function  $\theta(t, \varepsilon)$ , smoothly depending on  $t$ , such that, as  $\varepsilon \rightarrow 0$ ,

$$|\varepsilon d\theta/dt - 1| \rightarrow 0, \quad |x(t, \varepsilon) - \varphi_i(\theta(t, \varepsilon), Y_i(t))| \rightarrow 0,$$

$$|y(t, \varepsilon) - Y_i(t)| \rightarrow 0,$$

where  $i = 1$  for  $t_1 \leq t \leq t_2$ ;  $i = 2$  for  $t_2 < t \leq t_3$ .

The author takes this opportunity to express deep gratitude to L. S. Pontryagin and D. V. Anosov for posing the problem and for their attention to the present work.

Received  
16 VII 1969

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*Note: Figure translations are in progress. See original paper for figures.*

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