

# DOUBLE SUMS WITH A QUADRATIC CHARACTER

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**Abstract**

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## **DOUBLE SUMS WITH A QUADRATIC CHARACTER**

*(Presented by Academician I. M. Vinogradov, 23 X 1968)*

In the present article we formulate results on estimates of double sums with a quadratic character.

§ 1. Let  $\chi(m, n)$  denote  $\chi_n(m) = \left(\frac{\pm m}{n}\right)$ —the Jacobi symbol, or  $\chi_m(n) = \left(\frac{\pm m}{n}\right)$ —the Kronecker symbol, extended to the set of values of the arguments where these symbols are not defined by assigning the value zero.

We consider the sums

$$S_{ab}(YZ; MN) = \sum_{Y \leq n \leq Z} \sum_{M \leq m \leq N} a_n b_m \chi(m, n), \quad (1)$$

where  $a_n, b_m$  are arbitrary complex numbers;  $Y, Z, M, N$  are any numbers  $> 1$ . Sums of this kind arise in certain questions of the theory of quadratic forms, in questions concerning the distribution of divisors of quadratic fields, etc.

In the general case, for the indicated double sums (1) the following result holds.

**Theorem 1.**

$$\begin{aligned} S_{ab}(YZ; MN) = & \sum_{\substack{Y \leq n^2 \leq Z \\ (n, \gamma) = 1}} a_{n^2} \sum_{\substack{M \leq m \leq N \\ (m, n) = 1}} b_m \\ & + O\left(A \min[N^{1/2} Y^{-1/2} B_1 + N^{1/4} B_2; (NY^{-1/4} + N^{1/2} Z^{1/4}) B_3; \right. \\ & \left. \left( \sum_{Y \leq n \leq Z} \sum_{\substack{l=1 \\ (l, n) = 1}}^n \left| \sum_{\substack{M \leq m \leq N \\ M \equiv l \pmod{n}}} b_m - F \right|^2 \right)^{1/2} \right], \end{aligned} \quad (2)$$

where  $F$  is any quantity not depending on  $l$ ;  $\gamma = 2$  or  $1$ , according as  $\chi(m, n) \equiv \chi_n(m)$  or  $\chi_m(n)$ ;  $(r, k)$  is the greatest common divisor of  $r$  and  $k$ ;

$$A^2 = \ln^2 ZN \sum_{Y \leq n \leq Z} n|a_n|^2; \quad B_1^2 = \sum_{M \leq m \leq N} \frac{|b_m|^2}{m};$$

$$B_2^2 = \sum_{M \leq m \leq N} \frac{|b_m|^2}{\sqrt{m}}; \quad B_3^8 = \sum_{M \leq m \leq N} \frac{|b_m|^8}{m}.$$

Here we have no possibility to dwell on the applications for the sake of which this theorem was obtained. Therefore we shall explain the scheme of its application on a simple example.

In the generally accepted notation for number-theoretic functions, consider the sum

$$\Sigma \equiv \sum_{Y \leq n \leq Z} \sum_{1 < m \leq N} \mu(n) \chi_d(n) \tau_k^l(n) \tau_r(m) \chi(m, n). \quad (3)$$

For brevity, instead of  $O(1)$  and  $(1/\ln N)^{c_1}$  we shall write, respectively,  $C$ ,  $\ll$ , and  $L$ .

Replacing the terms in (3) by their moduli gives

$$\ll NZ \ln^C NZ. \quad (4)$$

Let us apply the theorem in the case

$$a_n = \mu(n) \chi_d(n) \tau_k^l(n); \quad b_m = \tau_r(m),$$

where

$$A \ll Z \ln^C Z; \quad B_\nu \ll \ln^C N.$$

Let  $Z > Y \geq N^2$ . Then, by virtue of the first term standing under the sign  $\min$ , we have

$$\Sigma \ll N^{1/2} Z \ln^C NZ,$$

and if in (3) the function  $\mu$  is removed, then the corresponding sum will also have a “main term” :

$$\sum_{\substack{Y < n^2 \leq Z \\ (n,r)=1}} \chi_d(n^2) \tau_k^l(n^2) \sum_{\substack{1 < m \leq N \\ (m,n)=1}} \tau_r(m) \ll NZ^{1/2} \ln^C NZ.$$

Consequently, for  $Y \geq N^2$  the sum (3) has an ideal saving by  $N^{1/2}L$ . On the interval ( $Y = N^{3/2}$ ;  $Z = N^2$ ), owing to the same term in (2), the saving will be by  $N^{1/4}L$ , while for  $Y \leq n \leq N^{3/2}$  the second of the remainder terms in (2) begins to operate, providing a saving, in comparison with the trivial estimate (4), by  $(Y^{-1/4} + N^{1/8})L$ .

Finally, on the interval  $1 < n \leq Y = N^\alpha$  the last term from (2) comes into play, and since the sums  $\sum \tau_r(m)$  are uniformly distributed over primitive progressions  $m \equiv l \pmod{n}$ ,  $m \leq N$ , with  $n \leq N^{1/r}$  (see, for example, (4)), this gives a saving on the interval  $(1; N^{1/r})$  by  $N^{1/2r}L$ .

Thus, in any case, for  $Z \geq N^\alpha$  a power saving is obtained for the sum (3).

In general, despite its generality, Theorem 1 substantially takes into account the interfering influence of the character and, for a wide class of sequences  $a_n, b_m$ , gives a good estimate of the sums (1). Often this estimate turns out to be simply an asymptotic expression with a power saving in the remainder term.

The idea of the proof of Theorem 1, as well as of the following Theorems 2–4, consists in making the most advantageous use of the Vinogradov–Pólya estimate (1) for character sums, and where this cannot be done, reducing the matter to estimates by the “large sieve” method. If the length of summation over  $m$  is close to  $n$ , then, as is known, the Vinogradov–Pólya estimate is close to ideal. The shorter the length of the sum over  $m$ , the worse the estimate, but by means of character reciprocity one can pass to long sums over  $n$ . However, for “very short” sums over  $n$ , such a reduction to long sums over  $m$ , as is seen from the example of the estimate of the sums (3), is, generally speaking, not suitable. But in that case the problem of estimating the sums (1) reduces to an expression typical for the “large sieve” method (see (3)), indicated last in formula (2).

§ 2. In applications one encounters sums of the form (1) in which one, or simultaneously both, of the sequences  $a_n, b_m$  are smooth. In such cases the estimates of the preceding theorem can be somewhat strengthened.

Let, further,

$$S_{a\lambda}^*(YZ; MN) = \sum_{M \leq d \leq N} \sum_{Y \leq n \leq Z} a_n \chi(\lambda_d, n).$$

**Theorem 2.** If  $\lambda_d$  runs through a sequence of numbers of a primitive arithmetic progression  $Dd + l$ ,  $1 \leq l \leq D$ , then, whatever ...

were complex numbers  $a_n$ ,

$$S_{a\lambda}^*(YZ; MN) = (N - M) \sum_{\substack{Y \leq n^2 \leq Z \\ (n, \gamma) = 1}} a_{n^2} \prod_{p|n} \left(1 - \frac{1}{p}\right) + \\ + O \left[ \ln^2 NZD \min(N^{3/2} Y^{-1/2} D + N^{1/2}; AZ^{1/2}) \right],$$

where  $A, \gamma$  are the same as in Theorem 1.

**Theorem 3.** Let  $\lambda_d$  run through some increasing subsequence of the natural numbers greater than 1; let  $a_x$  be a continuously differentiable (real or complex) function of  $x$ . Then

$$S_{a\lambda}^*(YZ; MN) = \sum_{\substack{Y \leq n^2 \leq Z \\ (n, \gamma) = 1}} a_{n^2} \sum_{r|n} \mu(r) \sum_{\substack{M \leq d \leq N \\ \lambda_d \equiv 0 \pmod{r}}} 1 + \\ + O \left( \min \left[ N \lambda_N^{1/2} \left( \int_Y^Z |a'_x| dx + a_Z \right) \ln \lambda_N; (NY^{-1/4} + N^{1/2} Z^{1/4}) A; \right. \right. \\ \left. \left. \left( A^2 \sum_{Y \leq n \leq Z} \sum_{\substack{l=1 \\ (l, n) = 1}} \left| \sum_{\substack{M \leq d \leq N \\ \lambda_d \equiv l \pmod{n}}} 1 - F \right| \right)^{1/2} \right] \right),$$

where  $F, A, \gamma$  are the same as before.

**Theorem 4.** If  $\lambda_d$  runs through the numbers of the primitive progression  $Dd + l$ ,  $1 \leq l \leq D$ ;  $a_x$  is a continuously differentiable (real or complex) function of  $x$ , then

$$S_{a\lambda}^*(YZ; MN) = (N - M) \sum_{\substack{Y \leq n^2 \leq Z \\ (n, \gamma) = 1}} a_{n^2} \prod_{p|n} \left( 1 - \frac{1}{p} \right) + \\ + O \left( \min \left[ AZ^{1/2}; N^{3/2} D^{1/2} \left( \int_Z^Y |a'_x| dx + a_Z \right) \ln DN \right] \right).$$

Let  $h(-d)$  denote the number of classes of purely radical positive quadratic forms of discriminant  $-d$ , where  $d > 1$ . Then a simple combination of Dirichlet's well-known exact formula for  $h(-d)$  with Theorem 4 gives:

$$\sum_{d \leq N} h(-d) = cN^{3/2} + O(N \ln N). \quad (5)$$

But according to the results of I. M. Vinogradov from (2), the second main term on the right in (5) is equal to  $-\frac{2}{\pi^2} N$ .

This example shows that in some cases the estimates of Theorem 4 differ from the actual values of the quantities being estimated by no more than  $\ln N$ .

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### References Cited

- <sup>1</sup> I. M. Vinogradov, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **27**, no. 1, 3 (1963).
- <sup>2</sup> I. M. Vinogradov, *Selected Works*, Publishing House of the Academy of Sciences of the USSR, 1952, p. 58.
- <sup>3</sup> K. Prachar, *Distribution of Prime Numbers*, Moscow, 1967.
- <sup>4</sup> A. F. Lavrik, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **30**, no. 2, 433 (1966).

*Note: Figure translations are in progress. See original paper for figures.*

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