Reports of the Academy of Sciences of the USSR

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1969. Volume 188, No. 4

UDC 519.46

MATHEMATICS

B. I. ROSENFELD

ON PRINCIPAL FIBERINGS WHOSE BASES SATISFY THE SECOND AXIOM OF COUNTABILITY

(Presented by Academician P. S. Aleksandrov on 28 II 1969)

In the paper of I. M. Gelfand and D. B. Fuks (^{1}) an extension was considered of the classical category of locally trivial principal G-fiberings, where G is a connected Lie subgroup of the group $GL(n, \mathbf{R})$. The objects of this category were principal G-fiberings with Hausdorff bases, locally trivial in the following sense: for each point x of the base of such a fibering there exists a mapping of the space $E^{\#}$ of the associated vector fibering into the space $\mathbf{R}^n$, linear on each fiber of the vector fibering and nondegenerate on the fiber over the point x. We emphasize that for fiberings with completely regular bases this notion of local triviality does not differ from the classical one, and that the local triviality of a G-fibering is defined depending not only on the group G, but also on the embedding $G \subset GL(n, \mathbf{R})$. In (^{1}), in this category, a subcategory was singled out that plays a role analogous to that of the universal G-fibering. It was also indicated there that if the category under consideration is narrowed by requiring the bases of the fiberings to satisfy the second axiom of countability, then there exists a fibering ξ_G such that for any fibering ξ of our category there is a mapping $\xi \to \xi_G$.

The fibering $(\xi_G)_p : \mathcal{E}_n \to \mathcal{S}_G$ is constructed as follows. Let $\overline{\mathcal{E}}$ be the product of a countable number of real lines in the Tikhonov topology. By $\mathcal{E}_n$ we denote the set of all nondegenerate n-frames in the linear space $\overline{\mathcal{E}}$. The group $GL(n, \mathbf{R})$ and its subgroup G act on the space $\mathcal{E}_n$ in the obvious way. We denote the quotient space $\mathcal{E}_n/G$ by $\mathcal{S}_G$; the projection $p : \mathcal{E}_n \to \mathcal{S}_G$ defines, obviously, a principal G-fibering, which is denoted by ξ_G.

All the constructions described carry over without change to the case when G is a Lie subgroup of the group $GL(n, \mathbf{C})$. The corresponding universal fibering will likewise be denoted by $(\xi_G)_p : \mathcal{E}_n \to \mathcal{S}_G$.

The aim of the present work is to show that the answers to the basic questions considered in (^{1}) change essentially under the above narrowing of the category. For example, it turns out that for a wide class of (noncompact) groups G the spaces $\mathcal{S}_G$ are paracompact, from which the traditional consequences are obtained.

Let us list the main results of the paper.

Denote by $\mathfrak{H}$ the class of such subgroups G of the groups $GL(n, \mathbf{C})$ for which the base $\mathcal{S}_G$ of the universal G-fibering is paracompact. Conditions sufficient for G to belong to this class are given by the following theorems:

Theorem 1. *Lie subgroups of the group $L(n, \mathbf{C})$ of matrices of order n whose determinant modulus is equal to one belong to the class $\mathfrak{H}$.*

Theorem 2. *If G is connected, $G \cap L(n, \mathbf{C})$ is a compact group, and in G there exists a matrix C with such eigenvalues λ_1 and λ_2 that $|\lambda_1| > 1,\ |\lambda_2| < 1$, then G belongs to $\mathfrak{H}$.*

The following theorems follow from the definition of the class $\mathfrak{H}$.

Using our narrowed category of G-bundles, we can, analogously to how this was done in (^{1}), define the groups $H^G_{\text{ag}}(G, \mathbf{R})$.

Theorem 3. If $G \in \mathfrak{S}$, then $H^q_{\text{alg}}(G, \mathbf{R}) = 0$ for $q > 0$.

Theorem 4. Let $G \in \mathfrak{S}$; then for any G-bundle $f : X \to Y$, the paracompactness of the space X implies the paracompactness of the space Y.

On the other hand, for many noncompact groups the base of the universal bundle has poor topological properties.

Put

$$\|G\| = \inf_{g \in G} \|g\|, \qquad \text{where } \|g\| = \sum_{i,j} |x_{ij}| \quad \text{for } g = (x_{ij}).$$

Theorem 5. If $\|G\| = 0$, then every continuous real-valued function on $\mathcal{P}_G$ is constant.

The condition $\|G\| = 0$ is satisfied, for example, for the group $GL(n, \mathbf{C})$. It turns out that for $GL(n, \mathbf{C})$ the groups $H^q_{\text{alg}}(G, \mathbf{R})$ are nontrivial and, moreover, the space $\mathcal{P}_{GL(n,\mathbf{C})}$ can be the base of a nontrivial Hilbert bundle.

Example. Let $H = L_2(\mathbf{C}^n)$ be the space of square-summable functions on $\mathbf{C}^n$. Consider the unitary representation of the group $GL(n, \mathbf{C})$ acting in H, which is given by the formula

$$gf(z) = |\det g|\, f(g^{-1}z),$$

where $g \in GL(n, \mathbf{C})$, $f(z) \in L_2(\mathbf{C}^n)$. With the help of Theorem 5 one can show that the bundle induced by this representation,

$$\mathcal{E}_n \times H \to \mathcal{E}_n \times H/G \simeq \mathcal{P}_{GL,\mathbf{C}},$$

with base $\mathcal{P}_{GL}$ and fiber H, is nontrivial.

Analogous results can also be obtained for real linear groups.

Proof of Theorem 1.

Case 1. The group $G = L(n, \mathbf{C})$.

Denote by A the space of all collections $\{x_{p_1,\dots,p_n}\}_{p_i=1,2,\dots}$, where $x_{p_1,\dots,p_n}$ are real numbers, endowed with the Tikhonov topology. The continuous mapping $f^* : \mathcal{E}_n \to A$, given by the formula

$$f^*(x) = \{(f^*x)_{p_1,\dots,p_n}\},$$

where

$$(f^*x)_{p_1,\dots,p_n} = \left|\det \begin{pmatrix} x_{1p_1} & \dots & x_{1p_n} \\ \dots & \dots & \dots \\ x_{np_1} & \dots & x_{np_n} \end{pmatrix}\right|$$

for $x = \{x_{ij}\}$ (recall that the points of the space $\mathcal{E}_n$ are n-frames in the space $\mathcal{E}$, and each point $x \in \mathcal{E}_n$ can be specified by a collection $\{x_{ij}\}_{i=1,\dots,n;\, j=1,2,\dots}$), is constant on the orbits of the group G and therefore defines a continuous mapping $f : \mathcal{P}_G \to A$. It is obvious that the mapping f takes distinct points to distinct points. We shall show that f is a topological embedding. For this purpose, for every point $y \in f(\mathcal{P}_G) \subset A$ we construct a continuous mapping h of a certain neighborhood of this point (in the space A) into $\mathcal{E}_n$ and show that the mapping $f \circ p \circ h|_{f(\mathcal{P}_G)}$ is the identity. Such a mapping can be defined by setting

$$h(\{y_{p_1,\dots,p_n}\}) = \{y_{ij}\},$$

where

$$y_{ij} = \begin{cases} y_{j,p_2^0,\dots,p_n^0}, & \text{for } i = 1, \\ \dfrac{y_{p_1^0,\dots,p_{i-1}^0,j,p_{i+1}^0,\dots,p_n^0}}{y_{p_1^0,\dots,p_n^0}}, & \text{for } i > 1 \end{cases}$$

(here $p_1^0, \dots, p_n^0$ is any collection such that

$$y_{p_1^0,\dots,p_n^0} \neq 0$$

).

As is known, A is a metric space. Consequently, $\mathcal{P}_G$ is also a metric space. Since the space $\mathcal{P}_G$ satisfies the second axiom of countability, it is paracompact.

Case 2. The group G is a Lie subgroup of the group $L(n, C)$. In this case there exists a smooth fibration $\mathcal{S}_G \to \mathcal{S}_{L(n,)}$ with fiber $L(n, C)/G$. The base of this fibration is paracompact, and the fiber is a finite-dimensional manifold. Consequently, $\mathcal{S}_G$ is also paracompact.

Proof of Theorem 2. Clearly, the group G is not contained in $L(n, C)$. Let $d : G \to \mathbb{R}$ be the homomorphism defined by the formula

$$d(g) = \ln|\det g|.$$

Denote by Γ the closed normal divisor $G \cap L(n, C)$. By hypothesis Γ is compact. There is an exact sequence

$$0 \to \Gamma \to G \to \mathbb{R} \to 0.$$

In an obvious way one constructs a homomorphism $\bar{d} : \mathbb{R} \to G$ such that

$$d\bar{d} : \mathbb{R} \to \mathbb{R}$$

is the identity mapping. It is clear that $\operatorname{Im} \bar{d}$ is a closed one-parameter subgroup of the group G. Its elements are the matrices $\exp(tA)$, where $t \in \mathbb{R}$, and A is some matrix. We denote this subgroup by B. We may assume that $\exp(A) = C$, where C is the matrix from the hypothesis of the theorem. Then the matrix A, in some basis, has the form:

$$A = \left(\begin{array}{ccc} \mu_1 & 0 & \\ 0 & \mu_2 & 0 \\ \hline 0 & & A' \end{array}\right); \qquad \exp(tA) = \left(\begin{array}{ccc} \exp(t\mu_1) & 0 & \\ 0 & \exp(t\mu_2) & 0 \\ \hline 0 & & \exp(tA') \end{array}\right),$$

where A' is some matrix and $\operatorname{Re} \mu_1 > 0$, $\operatorname{Re} \mu_2 < 0$. The universal fibration $\mathcal{E}_n \to \mathcal{S}_G$ decomposes into the composition

$$\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{S}_\Gamma \xrightarrow{B} \mathcal{S}_G,$$

where $\mathcal{S}_G$ is obtained after factorizing $\mathcal{S}_\Gamma$ by the action of the group B. We prove that the fibration

$$\mathcal{S}_\Gamma \xrightarrow{B} \mathcal{S}_G$$

is trivial.

For this purpose consider real continuous functions f_1, f_2 on $\mathcal{S}_\Gamma$, defined by the formulas

$$f_1(\Gamma\{x_{ij}\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\displaystyle\int_\Gamma |x_{1n}|\, d\Gamma}{1 + \displaystyle\int_\Gamma |x_{1n}|\, d\Gamma}; \qquad f_2(\Gamma\{x_{ij}\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\displaystyle\int_\Gamma |x_{2n}|\, d\Gamma}{1 + \displaystyle\int_\Gamma |x_{2n}|\, d\Gamma},$$

where $\Gamma\{x_{ij}\}$ is the orbit of the element $\{x_{ij}\} \in \mathcal{E}_n$.

On each orbit of the group $B = \{\exp(tA)\}$ the function f_1 increases monotonically, while the function f_2 decreases monotonically (as t increases). Moreover,

$$\lim_{t \to +\infty} f_1(\exp(tA)x) = a(x) > 0; \qquad \lim_{t \to -\infty} f_2(\exp(tA)x) = b(x) > 0;$$

$$\lim_{t \to -\infty} f_1 = \lim_{t \to +\infty} f_2 = 0.$$

The function $f = f_1 - f_2$ on an orbit of the group B increases monotonically from $-b(x)$ to $a(x)$. Therefore the set $f^{-1}(0) \subset \mathcal{S}_\Gamma$ defines a section surface of the fibration

$$\mathcal{S}_\Gamma \xrightarrow{B} \mathcal{S}_G.$$

Consequently, $\mathcal{S}_\Gamma$ is homeomorphic to $\mathcal{S}_G \times \mathbb{R}$. The space $\mathcal{S}_\Gamma$ is paracompact. Consequently, $\mathcal{S}_G$ is also paracompact. Theorem 2 is proved.

Theorem 3 is obvious.

Proof of Theorem 4. According to (^{1}), there exists a mapping of an arbitrary G-fibration $(\xi)f : X \to Y$ into the fibration $(\xi_G)p : \mathcal{E}_n \to \mathcal{S}_G$. Since $\mathcal{S}_G$ is paracompact, there exists a mapping of the fibration (ξ_G) into the classical universal fibration $(\xi^*_G)p_G : EG \to BG$, described in (^{2}). This latter decomposes into the composition

$$EG \to B\hat{G} \to BG,$$

where $\hat{G} \subset G$ is a maximal compact subgroup of the group G. The first arrow denotes a principal $\hat{G}$-fibration, and the second is a fibration having a section surface. It follows from this that our fibration ξ also decomposes into the composition

$$X \to \hat{X} \to Y.$$

Here the first arrow denotes a principal $\hat{G}$-fibration, and therefore from the paracompactness of X there follows the paracompactness

$\tilde{X}$; the second arrow denotes a bundle having a cross-section, and therefore Y is homeomorphic to a subspace of the space $\tilde{X}$ and, consequently, is also paracompact.

Proof of Theorem 5. Let h be a nonconstant continuous function on $\mathcal{P}_G$. Denote by f the continuous function on $\mathcal{E}_n$ defined by the formula $f(z) = h(pz)$ for $z \in \mathcal{E}_n$. There exist $x, y \in \mathcal{E}_n$ such that $f(x) \neq f(y)$; put $|f(x) - f(y)| = \varepsilon$. By the definition of the space $\mathcal{E}_n$, its points are n-tuples in $\mathcal{E}$, and each point $z \in \mathcal{E}_n$ can be specified by a set $\{z_{ij} \mid i = 1, \dots, n;\ j = 1, 2, \dots\}$. Let, for example, $x = \{x_{ij}\}$, $y = \{y_{ij}\}$.

For each natural number k choose a point $z^k = \{z^k_{ij}\} \in \mathcal{E}_n$ such that $z^k_{ij} = 0$ for $j \leq k$. For any element $g \in G$ put $g(z^k) = \{(gz)^k_{ij}\}$. Put $x(g)^k = \{x(g)^k_{ij}\}$, $y(g)^k = \{y(g)^k_{ij}\}$, where

$$x(g)^k_{ij} = \begin{cases} x_{ij}, & (j \leq k), \\ (gz)^k_{ij}, & (j > k); \end{cases} \qquad y(g)^k_{ij} = \begin{cases} y_{ij}, & (j \leq k), \\ (gz)^k_{ij}, & (j > k). \end{cases}$$

It is clear that, for sufficiently large k, the sets $x(g)^k,\ y(g)^k$ define elements of the space $\mathcal{E}_n$ and

$$|f(x) - f(x(g)^k)| < \varepsilon/4, \qquad |f(y) - f(y(g)^k)| < \varepsilon/4 \tag{1}$$

for every $g \in G$. Since $\|G\| = 0$, one can find such a $g_0 \in G$ that

$$|f(z^k) - f(g_0 x(g_0^{-1})^k)| < \varepsilon/4, \qquad |f(z^k) - f(g_0 y(g_0^{-1})^k)| < \varepsilon/4. \tag{2}$$

Since the function f is constant on the orbits of the group G, we have

$$f(g_0x(g_0^{-1})^k) = f(x(g_0^{-1})^k), \qquad f(g_0y(g_0^{-1})^k) = f(y(g_0^{-1})^k),$$

and from inequalities (1) with $g = g_0^{-1}$ and (2) we obtain $|f(x) - f(y)| < \varepsilon$, which contradicts the definition of the number ε.

In conclusion the author expresses deep gratitude to I. M. Gel' fand and D. B. Fuks for posing the problem and for their help in the work.

Faculty of Mechanics and Mathematics Moscow State University named after M. V. Lomonosov

Received 28 II 1969

REFERENCES

1. I. M. Gel' fand, D. B. Fuks, *Functional analysis and its applications*, **1**, no. 4 (1967).
2. N. Steenrod, *The Topology of Fibre Bundles*, Moscow, 1953.

Note: Figure translations are in progress. See original paper for figures.