ASYMPTOTICS OF AN AXISYMMETRIC SOLUTION

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Abstract

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ASYMPTOTICS OF AN AXISYMMETRIC SOLUTION

OF THE FLOW-PAST PROBLEM FOR THE NAVIER-STOKES EQUATIONS

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Let G be a domain in the space R^3, exterior with respect to a closed surface Σ. As is known, the flow-past problem for the Navier–Stokes equations consists in finding a solution of the system

$$\nu\Delta\mathbf{w} - \nabla p = (\mathbf{w}\cdot\nabla)\mathbf{w}, \qquad \operatorname{div}\mathbf{w} = 0 \tag{1}$$

in G, satisfying the conditions

$$\mathbf{w}\big|_{\Sigma} = 0, \qquad \mathbf{w}\to\mathbf{w}_0 = \text{const} \neq 0 \quad \text{as } |x| = r \to \infty. \tag{2}$$

Here $\mathbf{w} = \mathbf{w}(x)$, $p = p(x)$ are, respectively, the velocity and the pressure; $x(x_1, x_2, x_3)$ is a point of the space R^3; $\nu > 0$ is a constant.

Leray proved [1] (see also [2]) the existence of a solution of system (1) for which the Dirichlet integral is finite,

$$\int_G |\nabla\mathbf{w}|^2\,dx < \infty \tag{3}$$

and the first of conditions (2) is satisfied, while the condition at infinity is satisfied in a certain generalized sense. (Physically, condition (3) means that the given solution describes a flow of a viscous fluid in which the rate of dissipation of kinetic energy is finite.) We note that Leray' s theorem does not require any restrictions on the data of the problem. Subsequently [2,3] it was proved that solutions with finite Dirichlet integral satisfy the second of conditions (2), and their derivatives tend to zero at infinity. However, no additional information on the asymptotic behavior of solutions of problem (1)–(2) with finite Dirichlet integral has so far been obtained.

Finn [4] showed that considerably more can be said about the asymptotics of the solution of the flow-past problem if one assumes that the condition

$$|\mathbf{w}(x) - \mathbf{w}_0| < Cr^{-1/2-\varepsilon} \quad \text{as } r \to \infty \tag{4}$$

is satisfied with some positive constants ε, C. In particular, every solution of (1)–(2) satisfying (4) admits the representation

$$\mathbf{w}(x) = \mathbf{w}_0 + \mathbf{a}\mathcal{E}(x) + O(|\nabla\mathcal{E}|) \quad \text{as } r \to \infty \tag{5}$$

and an analogous representation for the pressure. Here $\mathbf{a}$ is a constant vector; $\mathcal{E}(x)$ is the fundamental tensor of the Oseen system corresponding to (1)–(2) [5]. It follows from this that there is a paraboloidal wake region in the direction of $\mathbf{w}_0$, inside which $|\mathbf{w} - \mathbf{w}_0| = O(r^{-1})$. Outside any circular cone with axis directed along $\mathbf{w}_0$, $|\mathbf{w} - \mathbf{w}_0| = O(r^{-2})$.

Let us observe that every solution of (1)–(2) satisfying (4), on the basis of (5), has a finite Dirichlet integral. However, in the class of functions satisfying (4) it is not possible to establish an existence theorem for a solution of the flow-past problem, except in the case of small Reynolds numbers [6]. In connection with this, Finn posed the question: does every solution of the flow-past problem for which (3) is satisfied also satisfy (4)? It turns out–

it turns out that in the special case of axisymmetric solutions this question should be answered in the affirmative.

Below it is assumed that Σ is a surface of revolution with axis parallel to the vector $\mathbf{w}_0$. We shall call a solution $\mathbf{w} = (w_\rho, w_\varphi, w_z), p$ of problem (1)–(2) axisymmetric if $w_\varphi = 0$, and w_ρ, w_z, p do not depend on φ. Here $\rho = \sqrt{x_1^2 + x_2^2}$, $\varphi = \operatorname{arctg}(x_2/x_1)$, $z = x_3$ are cylindrical coordinates and the axis Oz is the axis of rotation of Σ.

The following analogue of Leray' s theorem is valid: for every sufficiently smooth surface of revolution Σ there exists at least one axisymmetric solution of system (1) satisfying conditions (2)–(3).

The main result of the present paper is contained in the theorem:

Theorem. *Every axisymmetric solution of problem (1)–(2) with finite Dirichlet integral (3) satisfies inequality (4) and, consequently, admits the asymptotic representation (5).*

As a consequence of axisymmetry, in the representation (5) only the component a_z of the vector $\mathbf{a}$ is different from zero. The vector $\mathbf{a}$ has a simple physical meaning: it is equal (under a suitable normalization of the tensor $\mathcal{E}$) to the force with which the fluid flow acts on the body being flowed around ([4]).

The basic point in the proof of the theorem is obtaining an estimate of the rate of decrease at infinity of the vorticity $\vec{\omega}$ of problem (1)–(2). In view of axisymmetry,

$$\omega_\rho = \omega_z = 0, \qquad \omega_\varphi \equiv \partial w_\rho/\partial z - \partial w_z/\partial \rho = \omega(\rho, z).$$

Introduce the function $f = \omega/\rho$ and denote $\mathbf{u} = \mathbf{w} - \mathbf{w}_0$. By the definition of f, it follows from equations (1) that

$$\mathcal{L}f \equiv \nu \left(\frac{\partial^2 f}{\partial \rho^2} + \frac{3}{\rho}\frac{\partial f}{\partial \rho} + \frac{\partial^2 f}{\partial z^2} \right) - u_\rho \frac{\partial f}{\partial \rho} - (w_0 + u_z)\frac{\partial f}{\partial z} = 0 \tag{6}$$

in the domain Ω, the intersection of G with the half-plane $\varphi = 0$. Since the solution $\mathbf{w}, p$ satisfies (3), it follows, on the basis of (3), that $\omega, u_\rho, u_z \to 0$ as $\sqrt{\rho^2 + z^2} \equiv r \to \infty$. From the interior a priori estimates ([2]) it follows that in the domain Ω all derivatives of f entering into (6) exist and are continuous. Moreover, in a neighborhood of the singular line $\rho = 0$ the inequalities

$$\left|\frac{u_\rho}{\rho}\right|, \quad |f|, \quad \left|\frac{1}{\rho}\frac{\partial f}{\partial \rho}\right| \le M < \infty \quad \text{as } \rho \to 0 \tag{7}$$

hold uniformly in z.

To prove (7), note that the components $w_{x_1}, w_{x_2}, w_{x_3}$ of the vector $\mathbf{w}$ are, by virtue of the interior a priori estimates, infinitely differentiable functions of the Cartesian coordinates x_1, x_2, x_3 in the domain G. In combination with the axisymmetry and solenoidality of $\mathbf{w}$, this leads to (7). It follows from the estimates (7) that equation (6) is satisfied in the limit as $\rho \to 0$.

Arguments analogous to those just presented show that $\omega/\rho = f \to 0$ as $r \to \infty$. We shall now prove that, for fixed ν, w_0, for any α, $0 < \alpha < 1/2$, there exists an $R = R(\alpha)$ such that for $r \ge R$ the inequality

$$|f| \le C_1 r^{-3/2+\alpha} e^{qz} K_{3/2}(qr) \equiv C_1 v(\rho, z) \tag{8}$$

is fulfilled. Here $q = w_1(1 - \alpha/4)/2\nu$; $K_{3/2}(s)$ is the Macdonald function ([7]); $K_{3/2}(s) > 0$ for $0 < s < \infty$; $K_{3/2}(s) = \sqrt{\pi/2s}\, e^{-s}[1 + O(s^{-1})]$ as $s \to \infty$. The symbols C_k, $k = 1, 2, \ldots$, here and below denote positive constants. We note that for $\alpha = 0$ the function $v(\rho, z)$ satisfies equation (6) in the "Oseen approximation," i.e. for $u_\rho = u_z = 0$.

Let $\alpha \in (0, 1/2)$ be fixed. Computations show that in the domain $R < r < \infty$, where R is sufficiently large, the inequality $\mathcal{L}v < 0$ holds. In doing so one uses the asymptotic representations of the Macdonald function and of its derivative, the first of the inequalities (7), and also the fact that

$u_\varphi,\ u_z \to 0$ as $r \to \infty$. We now choose C_1 so that the functions $g_\pm = C_1 v \pm f$ are nonnegative for $r = R$. The functions $g_\pm$ satisfy, for $R < r < \infty$, the differential inequality $\mathcal{L} g_\pm \leqslant 0$ and, moreover, $g_\pm \to 0$ as $r \to \infty$. Hence we conclude that for $r > R$ the functions $g_\pm$ cannot attain a negative minimum. For interior points of the domain Ω this follows directly from the definition of $\mathcal{L}$, and for points of the z-axis from the definition of $\mathcal{L}$ and inequalities (7) for the function f and the analogous inequalities for the function v. Taking into account the nonnegativity of $g_\pm$ for $r = R$, we obtain that $g_\pm \geqslant 0$ for all $r \geqslant R$, i.e. inequality (8).

As a consequence of (8) we have the a priori estimate for ω

$$|\omega| \leqslant C_2 \rho \exp\left[-q\left(\sqrt{\rho^2+z^2}-z\right)\right] / (\rho^2+z^2)^{1-\alpha/2} \tag{9}$$

for $r \geqslant R$.

It remains to prove how (4) follows from (9). To this end we use the representation of the solenoidal vector $\mathbf{u} = \mathbf{w} - \mathbf{w}_0$ in terms of its rotor. Denote by Σ_R the sphere $|x| = R$; by G_R the exterior of Σ_R; and by $\mathbf{n}$ the unit normal vector to Σ_R. If $\operatorname{div}\mathbf{u} = 0$, $\operatorname{rot}\mathbf{u} = \vec{\omega}$ in G_R and $\mathbf{u} \to 0$ as $r \to \infty$, then, according to [8], the representation

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2, \qquad \mathbf{u}_1 = \operatorname{rot}\vec{\psi}, \qquad \mathbf{u}_2 = \nabla\lambda, \tag{10}$$

holds, where

$$\vec{\psi}(x) = \frac{1}{4\pi}\int_{G_R} \frac{\vec{\omega}(y)}{|x-y|}\,dy, \tag{11}$$

and $\lambda(x)$ is the solution of the Neumann problem

$$\Delta\lambda = 0, \qquad x \in G_R, \qquad \partial\lambda/\partial n\big|_{\Sigma_R} = \mathbf{n}\cdot(\mathbf{u}-\mathbf{u}_1)\big|_{\Sigma_R}.$$

It is easy to see that for $\mathbf{w}_2 = \mathbf{u}_2 + \mathbf{w}_0$ condition (4) is satisfied with $\varepsilon = 3/2$. For $\mathbf{u}_1$, by virtue of (10)–(11), we obtain

$$|\mathbf{u}_1| \leqslant \frac{\sqrt{3}}{2\pi}\int_{G_R} \frac{|\vec{\omega}(y)|}{|x-y|^2}\,dy. \tag{12}$$

In estimating the last integral it is convenient to pass to cylindrical coordinates ρ, φ, z and, using the fact that $\vec{\omega} = (0, \omega, 0)$ does not depend on φ, to carry out the integration with respect to φ. Taking into account that for all $\rho, \xi > 0$, $z \neq \eta$,

$$\int_0^{2\pi} \frac{\rho\,d\varphi}{\rho^2+\xi^2-2\rho\xi\cos\varphi+(z-\eta)^2} \leqslant \frac{\pi}{\sqrt{(\rho-\xi)^2+(z-\eta)^2}},$$

and applying inequality (9), from (12) we find the estimate

$$|\mathbf{u}_1| \leqslant C_3 \iint_{\xi^2+\eta^2 \geqslant R^2} \frac{\xi\exp\left[-q\left(\sqrt{\xi^2+\eta^2}-\eta\right)\right]\,d\xi\,d\eta}{(\xi^2+\eta^2)^{1-\alpha/2}\sqrt{(\rho-\xi)^2+(z-\eta)^2}} \leqslant \frac{C_4 \ln r}{r^{1-\alpha}}$$

for $r = \sqrt{\rho^2 + z^2} \geqslant R$. In view of the arbitrariness of $\alpha \in (0, 1/2)$, we conclude from this that (4) is satisfied for $\mathbf{w} = \mathbf{u}_1 + \mathbf{w}_2$. The theorem is proved.

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References

1. J. Leray, J. Math. pures et appl., S. IX, **12**, 1 (1933).

2. O. A. Ladyzhenskaya, *Mathematical Problems of the Dynamics of a Viscous Incompressible Fluid*, Moscow, 1961.

3. R. Finn, Arch. Rat. Mech. and Anal., **3**, No. 5, 381 (1959).

4. R. Finn, Bull. Math. Soc. Sci. Math. Phys. de la R.P.R., **3** (53), No. 4, 387 (1959).

5. C. W. Oseen, *Neuere Methoden und Ergebnisse in der Hydrodynamik*, Leipzig, 1927.

6. R. Finn, Arch. Rat. Mech. and Analysis, **19**, No. 5, 363 (1965).

7. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Part 1, Moscow, 1949.

8. N. E. Kochin, *Vector Calculus and the Elements of Tensor Calculus*, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.