A SOLUTION FOR A CLASS OF GAMES WITH EMPTY CORE

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 518.9

MATHEMATICS

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A SOLUTION FOR A CLASS OF GAMES WITH EMPTY CORE

(Presented by Academician Yu. V. Linnik on 17 V 1968)

Consider a cooperative game $\Gamma = \langle I, v \rangle$, where $I = \{1, \dots, n\}$ is the set of players, and $v(S)$ is the characteristic function of the game. The set of imputations A is defined as

$$A = \left\{ x = (x_1, \dots, x_n) : x_i \geqslant v(i), \quad \sum_{i=1}^{n} x_i = v(I_n) \right\}.$$

On A a domination relation is defined: $x \succ y$, if there exists such an $S \subset I$ that $x_i > y_i,\ i \in S$, and

$$\sum_{i \in S} x_i \leqslant v(S).$$

A solution V is a set of imputations having the following properties:

1) internal stability: it cannot be that $x \succ y$, if $x, y \in V$;

2) external stability: whatever $z \notin V$, there exists such an $x \in V$ that $x \succ z$.

The core is the set:

$$U = \left\{ x = (x_1, \dots, x_n) \in A : \sum_{i \in S} x_i \geqslant v(s) \text{ for every } s \subset I \right\}.$$

We shall say that a game has property v if it has a solution, property cv if its solution coincides with the core, and property $\overline{cv}$ if this solution has the following additional property:

whatever $y \notin V$, if $\sum_S y_i < v(S)$, then there exists such an $x \in V$ that $x \succ y$ by the coalition S. Let us note that in all cases known to us where the solution coincides with the core, property $\overline{cv}$ follows from cv.

For any $R \subset I$ define a game $\Gamma_R(a_0, a_S)$ with set of players R and characteristic function $u(S)$, $S \subset R$:

$$u(R) = a_0, \quad 0 \leqslant a_0 \leqslant 1; \tag{1}$$

$$u(S) = a_S, \quad v(S) \leqslant a_S \leqslant \max_{T \subset I-R}[v(S \cup T) - v(T)] = \overline{a}_S.$$

Consider a partition of the set I into sets M and N, and let $v(M) + v(N) \geqslant 1$ (if $v(M) + v(N) > 1$, then the core is obviously empty). Construct $\Gamma_M(a_0, a_S)$ and $\Gamma_N(\beta_0, \beta_T)$.

Call the collections $\{a_0, a_S\}$ and $\{\beta_0, \beta_T\}$ conjugate if: 1) $a_0 + \beta_0 = 1$; 2) $a_S + \beta_T \geqslant v(S \cup T)$, $\quad S \subset M,\ T \subset N$.

Lemma. *Whatever the collection* $\{a_0, a_S\}$ $(S \subset M)$ *satisfying (1), there exists a conjugate collection* $\{\beta_0, \beta_T\}$ $(T \subset N)$.

Proof. Take $\beta_0 = 1 - a_0$,

$$\beta_T = \overline{\beta}_T = \max_{S \subset N}[v(S \cup T) - v(S)].$$

Then $\beta_T \geqslant v(T \cup S) - v(S)$, $S \subset M$,

$$\alpha_S + \beta_T \geqslant v(S) + v(T \cup S) - v(S) = v(T \cup S).$$

Let $V_M(a_0, \alpha_S)$ be a solution of $\Gamma_M(a_0, \alpha_S)$, and $V_N(\beta_0, \beta_T)$ a solution of $\Gamma_N(\beta_0, \beta_T)$; then by $V(a_0) = V_M(a_0, \alpha_S) \mathbin{\hat{\times}} V_N(\beta_0, \beta_T)$ we denote the set of all such $x \in A$ that

$$x_M \in V_M(a_0, \alpha'_S), \qquad x_N \in V_N(\beta_0, \beta'_T)$$

(x_M and x_N are the projections of x onto M and N, respectively), and the collections $\{a_0, \alpha'_S\}$ and $\{\beta_0, \beta'_T\}$ are conjugate.

Theorem 1. If in a game Γ I is divided into two such coalitions M and N (each with more than one player) that 1) $v(M) + v(N) \geqslant 1$; 2) all games $\Gamma_M(a_0, \alpha'_S)$ and $\Gamma_N(\beta_0, \beta'_T)$, for all conjugate collections $\{a_0, \alpha'_S\}$ and $\{\beta_0, \beta'_T\}$, where $1 - v(N) \leqslant a_0 \leqslant v(M)$, have the property $\overline{cv}$, then

$$V(a_0) = V_M(a_0, \alpha_S) \mathbin{\hat{\times}} V_N(\beta_0, \beta_T)$$

is a solution of Γ for all a_0 from the interval $1 - v(N) \leqslant a_0 \leqslant v(M)$.

Proof. The internal stability of $V(a_0)$ follows from the fact that, for $S \supseteq M$ or $S \supseteq N$, domination cannot occur, while for the remaining S and $x \in V(a_0)$

$$\sum_S x_i \geqslant \alpha(S \cap M) + \beta(S \cap N) \geqslant v(S).$$

We now prove the external stability of $V(a_0)$. Consider two cases:

$$1)\ \sum_M y_i = a_0;$$

$$2)\ \sum_M y_i < a_0 \quad \text{or} \quad \sum_N y_i < 1 - a_0.$$

Consider the first case. It is easy to show that, if $y \notin V(a_0)$, then there exists a coalition R such that

$$\sum_R y_i < v(R). \tag{2}$$

If, moreover, there exists $R \subset M$ (or $R \subset N$), then by the property $\overline{cv}$ there exists such an $x_M \in V_M(a_0, v(S))$ that $x_M > y_M$ (on R), and by the lemma there exists a collection $\{1 - a_0, \beta'_T\}$ conjugate with $\{a_0, v(S)\}$; then, if $x_N \in V_N(1 - a_0, \beta'_T)$, then $x = (x_M, x_N) > y$.

Suppose now that, for any R satisfying (2), $R \cap M \neq \Lambda$, $R \cap N \neq \Lambda$. Denote by R_0 such one of the coalitions R for which there is no coalition $R \subset R_0$ satisfying property (2).

Put

$$\alpha'_{R_0 \cap M} = \sum_{R_0 \cap M} y_i + \varepsilon_1; \qquad \alpha'_S = \sum_S y_i, \quad S \subset R_0 \cap M; \qquad \alpha'_S = \bar{\alpha}_S$$

for the remaining $S \subset M$.

$$\beta'_{R_0 \cap N} = \sum_{R_0 \cap N} y_i + \varepsilon_2; \qquad \beta'_T = \sum_T y_i, \quad T \subset R_0 \cap N; \qquad \beta'_T = \bar{\beta}_T$$

for the remaining $T \subset N$,

where

$$\varepsilon_1 + \varepsilon_2 = v(R_0) - \sum_{R_0} y_i, \qquad \varepsilon_1 > 0, \quad \varepsilon_2 > 0.$$

Since $\sum_S y_i \geqslant v(S)$, $S \underset{\neq}{\subset} R_0$, it is not hard to see that the sets $\{\alpha_0, \alpha'_S\}$ and $\{1-\alpha_0, \beta'_T\}$ are conjugate and, for R_0,

$$\alpha'_{R_0 \cap M} + \beta'_{R_0 \cap N} = v(R_0).$$

If $|R_0 \cap M| > 1$ and $|R_0 \cap N| > 1$, then by the property $\overline{cv}$ there exist $x_M \in V_M(\alpha_0, \alpha'_S)$ and $x_N \in V_N(1-\alpha_0, \beta'_T)$ such that $x_M \succ y_M$ on $R_0 \cap M$ and $x_N \succ y_N$ on $R_0 \cap N$, and hence $x = (x_M, x_N) \succ y$ on R_0.

If, for example, $R_0 \cap M = \{i_0\}$, then, by Theorem 4.1 of [(2)], in the solution (core) there exists an imputation $x_M \in V_M(\alpha_0, \alpha'_S)$ such that the component $(x_M)_{i_0}$ is equal to $\alpha(i_0)$, and the proof is preserved for this case as well.

Let now case 2 occur, and let $\sum_M y_i < \alpha_0$; consider

$$z_M = (z_M^{(i)}): \quad z_M^{(i)} = y_i + \varepsilon, \quad i \in M, \quad \sum_{i \in M} z_M^{(i)} = \alpha_0.$$

Either $z_M \in V_M(\alpha_0, v(S))$. Or there exists $x_M \in V_M(\alpha_0, v(S))$, $x_M \succ z_M$; then in the first case $z = (z_M, z_N) \succ y$ on M, and in the second $x = (x_M, x_N) \succ y$, where z_N, x_N are constructed as in the proof of the first case.

Consider examples.

Example 1. If for a four-person game there exists $0 \leqslant \alpha_0 \leqslant 1$ such that

$$v(1,2) \geqslant \alpha_0, \quad v(3,4) \geqslant 1-\alpha_0; \quad v(i,3) + v(i,4) \leqslant$$

$$\leqslant 1-\alpha_0,\ i = 1,2; \quad v(1,j) + v(2,j) \leqslant \alpha_0,\ j = 3,4.$$

then the game has a solution lying in the plane $x_1 + x_2 = \alpha_0$; note that no restrictions are imposed on the values $v(S)$ for $|S| = 3$.

It is known that a three-person game has the property cv ($\overline{cv}$) if and only if $v(i,j) + v(i,k) \leqslant v(I)$, $\{i,j,k\} \subset \{1,2,3\}$. It can also be shown that, if a three-person game with characteristic function $v(S)$ satisfies the condition cv ($\overline{cv}$), then the game with $v'(S) = v(S)$, $|S| > 1$, $v'(i) = a_i$, where $a_i \leqslant v(I - \{i\})$, $i = 1,2,3$, also has the property cv ($\overline{cv}$). Using this, we have:

Example 2. If for a five-person game

$$v(1,2,3) \geqslant \alpha_0; \quad v(4,5) \geqslant 1-\alpha_0; \quad \alpha_0 - \frac{1}{2} \leqslant v(i,j) \leqslant \alpha_0/2;$$

$$v(i,j,l) \leqslant \alpha_0/2; \quad v(i,l) \leqslant (1-\alpha_0)/2; \quad \{i,j\} \subset \{1,2,3\}; \quad l = 4,5,$$

where $0 \leqslant \alpha_0 \leqslant 1$; the remaining $v(S)$ are arbitrary, then such a game has a solution lying in the hyperplane $x_1 + x_2 + x_3 = \alpha_0$.

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Received
7 V 1968

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2. O. N. Bondareva, *Problems of Cybernetics*, vol. 10 (1963).

Note: Figure translations are in progress. See original paper for figures.