ASYMPTOTICS OF CERTAIN SUBSEQUENCES OF EIGENVALUES OF BOUNDARY-VALUE PROBLEMS FOR THE HELMHOLTZ EQUATION IN THE MULTIDIMENSIONAL CASE

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Abstract

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ASYMPTOTICS OF CERTAIN SUBSEQUENCES OF EIGENVALUES OF BOUNDARY-VALUE PROBLEMS FOR THE HELMHOLTZ EQUATION IN THE MULTIDIMENSIONAL CASE

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1. Let Ω be a domain of Euclidean space R^{n+1}, bounded by a sufficiently smooth closed surface Σ. We consider problems for the Helmholtz equation

$$[\Delta + \omega^2 C^{-2}(M)]U(M) = 0, \qquad M \in \Omega; \tag{1}$$

$$\text{I. } U|_\Sigma = 0 \quad \left(\text{or II. } \frac{\partial}{\partial n} U|_\Sigma = 0\right), \tag{2}$$

where Δ is the Laplace operator in R^{n+1}; $C(M) > 0$ is a sufficiently smooth function in Ω; ω is a real parameter.

The present note is devoted to constructing, by the ray method, asymptotics of such subsequences of eigenvalues of problem (1), (2) to which there correspond eigenfunctions concentrated in a neighborhood of certain one-dimensional cycles in Ω. Asymptotically, as $\omega \to \infty$, within the framework of the ray method the solution of equation (1) is sought in the form $U(M) \sim [u + O(1/\omega)] \exp(i\omega\Phi)$, where u and Φ satisfy the equations

$$(\nabla\Phi)^2 = C^{-2}(M), \tag{3}$$

$$2(\nabla u, \nabla\Phi) + u\Delta\Phi = 0. \tag{4}$$

2. Consider in Ω a closed curvilinear N-gon L_N, whose sides are formed by smooth curves and whose vertices belong to Σ. We shall call the polygon L_N an extremal one-dimensional cycle in Ω if the first variation $\delta I = 0$ on L_N, where $I = \int C^{-1}(M)\, d\sigma$ is the functional of geometrical optics ($d\sigma$ is an element of length in R^{n+1}).

In what follows it is assumed that, if the extremal cycle under consideration in Ω is stable in the first approximation ([1,2]), then there exists the desired subsequence of eigenvalues of problem (1), (2), to which there correspond eigenfunctions concentrated in a neighborhood of this cycle, i.e. essentially different from zero in some neighborhood of the cycle and decreasing sufficiently rapidly when moving away from it.

3. Let $\mathbf{r} = \mathbf{r}(s)$, $\mathbf{r} \in R^{n+1}$, be an extremal cycle in Ω with vertices $\mathbf{r}(s_1), \dots, \mathbf{r}(s_N) = \mathbf{r}(0)$; s is arc length along the cycle. In each plane normal to the cycle, introduce the radius vector $\mathbf{q}(s) \in R^n$ so that along the cycle $\mathbf{q}(s) \equiv 0$. Denote the components of $\mathbf{q}(s)$ by q_j, $j = 1, \dots, n$. The conjugate momenta are defined by the formula

$$p_j = C^{-1}(s, q_j) \frac{\partial}{\partial \dot{q}_j} \frac{d\sigma}{ds},$$

where $\dot{q}_j = \frac{d}{ds} q_j$. We expand the Hamiltonian function $\mathcal{H}(s; q_j; p_j)$, corresponding to the functional I, in a neighborhood of the cycle into the series

$$\mathcal{H}(s; q_j; p_j) = -\frac{1}{C(s,0)} + \mathcal{H}_2(s; q_j; p_j) + \mathcal{H}_3(s; q_j; p_j) + \dots, \tag{5}$$

where $\mathcal{H}_k(s; q_j, p_j)$ is a homogeneous polynomial of degree k in the canonical variables with coefficients depending on s. In the first approximation, discarding in the expansion (5) the terms of order $\mathcal{H}_k$, $k \geqslant 3$, we replace I by the functional I_0:

$$I_0 = \int \left\{ \sum_{j=1}^{n} p_j \, dq_j + \left[\frac{1}{C(s,0)} - \mathcal{H}_2(s; q_j, p_j) \right] ds \right\}, \tag{6}$$

for which the canonical system of equations has the form

$$\frac{d}{ds} q_j = \frac{\partial \mathcal{H}_2}{\partial p_j}, \qquad \frac{d}{ds} p_j = -\frac{\partial \mathcal{H}_2}{\partial q_j}, \qquad j = 1, 2, \dots, n. \tag{7}$$

A solution of the system (7)

$$\vec{\chi}^{\,k}(s) = (q_1^{(k)}(s), \dots, q_n^{(k)}(s), p_1^{(k)}(s), \dots, p_n^{(k)}(s)),$$

defined on the k-th ($k = 1, 2, \dots, N$) side L_N, i.e. for $s_{k-1} \leqslant s \leqslant s_k$, will be called a ray. Let the $2n$ vectors $Z_t^{(k)}(s)$, $t = 1, \dots, 2n$, form on the k-th side L_N a fundamental system of solutions of the equations (7). With the aid of the matrix

$$W^{(k)}(s) = \| Z_1^{(k)}(s), \dots, Z_{2n}^{(k)}(s) \|$$

the ray is defined by the formula

$$\vec{\chi}^{(k)}(s) = W^{(k)}(s)\mathbf{A}^{(k)}$$

through the specification of a $2n$-dimensional vector $\mathbf{A}^{(k)}$ independent of s.

Upon reflection of the ray from the boundary Σ in a neighborhood of the k-th vertex L_N we have:

$$\mathbf{A}^{(k+1)} = \left[W^{(k+1)}(s_k)\right]^{-1} \cdot \gamma_k \cdot W^{(k)}(s_k) \cdot \mathbf{A}^{(k)} + O(\|\vec{\chi}\|^2), \tag{8}$$

where the reflection matrices γ_k of order $2n$ are determined by the local properties of Σ at the point $\mathbf{r}(s_k)$, $\det\gamma_k = 1$. Let $\mathbf{A}_0^{(k)}$ specify some initial ray, and let $\mathbf{A}_1^{(k)}$ be the ray arising after a single "traversal" of the initial ray along the cycle; then, in the first approximation, discarding the correction terms in formula (8), we obtain

$$\mathbf{A}_1^{(k)} = \Gamma_k \mathbf{A}_0^{(k)},$$

and after m traversals

$$\mathbf{A}_m^{(k)} = \Gamma_k^m \mathbf{A}_0^{(k)},$$

with $\det\Gamma_k = 1$ by virtue of the existence of the integral invariant

$$\int \delta q_1 \dots \delta q_n \delta p_1 \dots \delta p_n$$

for the canonical system (7).

An extremal cycle is called stable in the first approximation (1) if

$$\|\mathbf{A}_m^{(k)}\| < M < \infty$$

as $m \to \infty$. Let λ_t, $t = 1, \dots, 2n$, be the eigenvalues of the matrix Γ_k.* The conditions for stability of the cycle in the first approximation are that $|\lambda_t| = 1$, and the elementary divisors of Γ_k are simple.

4. For cycles stable in the first approximation there exist $2n$ linearly independent Floquet solutions of the equations (7), $X_t(s)$, such that

$$X_j(s+s_N) = e^{i\varphi_j} X_j(s), \qquad X_{n+j}(s+s_N) = e^{-i\varphi_j} X_{n+j}(s),^{**} \qquad j = 1, \dots, n.$$

We divide the Floquet solutions into two groups of n each so that, for the solutions of the first group,

$$W(X_j, \overline{X}_t) \equiv \sum_{k=1}^{n} (q_k^{(j)}(s) \cdot \overline{p}_k^{(t)}(s) - p_k^{(j)}(s) \cdot \overline{q}_k^{(t)}(s)) = -i\delta_{jt}, \qquad j, t = 1, \dots, n,$$

where δ_{jt} is the Kronecker symbol; $\overline{X}_t$ is the vector complex-conjugate to the vector X_t. At the same time, for the Floquet solutions of the second group

$$W(X_{n+j}, \overline{X}_{n+t}) = +i\delta_{jt}.$$

Introduce normal coordinates [(4)] Q_j, P_j by the formula

$$\vec{\chi} = \sum_{j=1}^{n} (\mathbf{B}_j(s)Q_j + \mathbf{B}_{n+j}(s)P_j).$$

The functions $\mathbf{B}_t(s)$, periodic on the cycle, have the form

$$\mathbf{B}_j(s) = e^{-i\mu_j s} X_j(s), \qquad \mathbf{B}_{n+j}(s) = e^{i\mu_j s} X_{n+j}(s), \qquad \mu_j = \varphi_j / s_N.$$

In nor-

* The characteristic equation of the matrix Γ_k is reciprocal (see, for example, (3)) and therefore reduces to an equation of degree n.

** By these formulas the phases φ_j of the eigenvalues of the matrix Γ_k are determined uniquely as the increment of $\arg X_j$ on the cycle.

normal coordinates

$$I_0 = \int \left\{ -i \sum_{j=1}^{n} P_j \, dQ_j + \left[\frac{1}{C(s,0)} - \sum_{j=1}^{n} \mu_{jP_{jQ}} j \right] ds \right\},$$

and the ray is described by the functions $Q_j(s) = Q_j(0)e^{i\mu_{js}}$, $P_j(s) = P_j(0)e^{-i\mu_{js}}$, continuous on the cycle. Here $Q_j(0) = \overline{P_j(0)}$ are constants of integration.

5. The set of rays whose constants of integration satisfy the conditions $Q_j(0)P_j(0) = \varepsilon_j^2$, $j = 1, \dots, n$, where ε_j are fixed real parameters, forms in the space $R^{2n} \times S^1$* an invariant manifold T_ε, since for all s

$$Q_j(s)P_j(s) = Q_j(0)P_j(0) = \varepsilon_j^2, \qquad j = 1, \dots, n. \tag{9}$$

The section of T_ε by the Euclidean plane R^{2n}, $s = \text{const}$, is the direct product of n circles (9); therefore topologically T_ε is described as the direct product of $(n+1)$ circles $S^1 \times S^1 \times \dots \times S^1$. The manifolds T_ε are Lagrangian; therefore the solution of the eikonal equation (3) is found by integrating over T_ε the total differential

$$d\Phi = -i \sum_{j=1}^{n} P_j \, dQ_j + \left[\frac{1}{C(s,0)} - \sum_{j=1}^{n} \mu_{jP_{jQ}} j \right] ds.$$

6. The asymptotics of the eigenvalues of the problem (1), (2) within the framework of the ray method is found from the "quantization conditions" ([5,6]) following from the requirement that the function $ue^{i\omega\Phi}$ be single-valued on the invariant manifold T_ε. The one-dimensional homology group $H_1(T_\varepsilon)$ of the manifold T_ε is the direct product of $(n+1)$ free cyclic groups and has $(n+1)$ generators l_k, $k = 0, 1, \dots, n$. In the case of boundary-value problem I, the "quantization conditions," which constitute a system of $(n+1)$ equations with respect to ω and the n parameters ε_j, have the form

$$\omega \oint_{l_k} d\Phi = 2\pi m_k + \pi m_k' + \frac{\pi}{2} m_k'', \qquad k = 0, 1, \dots, n, \tag{10}$$

where m_k is an integer; m_k' and m_k'' are, respectively, the "reflection"and "caustic" intersection indices, i.e. the Kronecker intersection indices (see also ([6,10])) of the oriented basis cycles l_k, respectively, with the boundary Σ and the caustic. As n basis cycles l_j, $j = 1, \dots, n$, we take the circles (9) in the plane $s = \text{const}$; for them $m_j' = 0$, $m_j'' = 2$. The cycle l_0 is fixed by the conditions $Q_j = \text{const}$, $0 \le s \le s_N$; for it $m_0' = N$, $m_0'' = 0$.

From equations (10) there follows a formula for the eigenvalues $\omega_{[m]}$

$$\omega_{[m]} \int_0^{s_N} \frac{ds}{C(s, 0)} = 2\pi(m_0 + {}^1\!/_2 N) + \sum_{j=1}^{n} (m_j + {}^1\!/_2)\varphi_j, \tag{11}$$

where $m_c \gg 1$; φ_j are the Floquet exponents of the first group of Floquet solutions.

In the case of boundary-value problem II, in formula (11) the term ${}^1\!/_2 N$ should be omitted, since in this case the term $\pi m_k'$ is absent in equations (10).

Formula (11), obtained on the basis of the linear system (7), coincides with the formula found by the parabolic-equation method ([7,8]).

* The one-dimensional cycles under consideration are homeomorphic to the circle S^1, since the points of self-intersection L_N, if there are any, should, in view of the linearity of the problem (1), (2), be regarded as lying on different copies of the domain Ω.

7. Using "Birkhoff series"[9], one can, within the framework of the ray method, successively take into account further terms in the expansion (5), provided that φ_j and 2π are linearly independent over the ring of integers. In this case, at each step, in a neighborhood of a cycle stable in the first approximation, we shall have a continuous family of invariant Lagrangian manifolds with the same topology as in the first approximation. The correction terms to formula (11) that arise in this way are homogeneous polynomials in $(m_j + 1/2)$ of the second, etc., degrees. There are grounds,

however, for believing (see [11], where, in a special case, the next terms in the formula for the eigenvalues were obtained) that the corrections obtained by the ray method are meaningful only for $m_0 \gg m_j \gg 1$.

The proposed scheme admits a generalization to the case of more general elliptic operators.

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Note: Figure translations are in progress. See original paper for figures.