ON HÖLDER CONTINUITY OF GENERALIZED SOLUTIONS OF ELLIPTIC EQUATIONS WITH COEFFICIENTS FROM SPACES WITH MIXED NORM

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.964

MATHEMATICS

A. Kh. GUDIEV

ON HÖLDER CONTINUITY OF GENERALIZED SOLUTIONS OF ELLIPTIC EQUATIONS WITH COEFFICIENTS FROM SPACES WITH MIXED NORM

(Presented by Academician S. L. Sobolev on 15 VII 1968)

Consider the equation

$$\frac{\partial}{\partial x_i}(a_{ij}u_{x_j} + a_i u + f_i) + b_i u_{x_i} + au = f \tag{1}$$

under the condition that the leading coefficients $a_{ij}(x)$ are bounded and the equation is strictly elliptic, i.e., under the condition

$$\nu\xi_i\xi_i \leq a_{ij}\xi_i\xi_j \leq \mu\xi_i\xi_i; \qquad \nu, \mu = \text{const} > 0. \tag{2}$$

With respect to the lower-order coefficients and the free terms we shall assume

$$\|a_i^2;\, b_i^2;\, f_i^2;\, a,\, f\|_{L_{(r_1,r_2)}(D)} \leq \mu; \tag{3}$$

$$(r_1, r_2) \in \Omega_2^{(s)}. \tag{4}$$

(For the definitions of the space $L_{(r_1,r_2)}(D)$ and the set $\Omega_2^{(s)}$, see (^{8}).)

Let us note that, when condition (4) is satisfied, the spaces $L_{(r_1,r_2)}(D)$ contain elements $f(x)$ belonging to $L_{1+\alpha}$ (α is a sufficiently small positive number), but not belonging to the space $L_{n/2}(D)$.

In the present work it is proved that every generalized solution $u(x)$ from $W_2^1(D)$ of equation (1), when conditions (2), (3), (4) are fulfilled, belongs to the class $C_{0,\alpha}$.

In the particular case $r_1 = r_2 = q/2,\ q > n$, i.e., when

$$\|a_i^2;\, b_i^2;\, f_i^2;\, a,\, f\|_{L_{q/2}(D)} \leq \mu, \tag{5}$$

the membership of a generalized solution $u(x)$ from $W_2^1(D)$ in $C_{0,\alpha}$ had previously been established in a number of works ($^{3-5}$). We note that the study of the smoothness of a generalized solution in the case $r_1 = r_2 = q/2,\ q > n$, and in the case considered by us, differ substantially.

The works mentioned above in one way or another rely on Sobolev imbedding theorems (^{1}), which are not applicable in our case. This is also understandable, since the spaces $L_{(r_1,r_2)}(D)$ contain functions belonging to $L_p,\ 1 < p \leq n/2$.

Thanks to the introduction of spaces with mixed norm $L_{(r_1,r_2)}(D)$ and the establishment for them of imbedding theorems and certain integral estimates (^{9}), and also thanks to a generalization of De Giorgi' s lemma and of a number of results of O. A. Ladyzhenskaya and N. N. Ural' tseva, it has been possible to study membership in the Hölder class of generalized solutions of a broad class of elliptic equations of the form (1).

We shall state the results obtained. First let us introduce some notation. Let D be an n-dimensional bounded domain belonging to E^n; let s be a natural number less than n; let K_ρ be a ball in E^n of radius ρ; $D_\rho = \overline{K}_\rho \cap D$; $\operatorname{osc}\{u(x); D\}$ the oscillation of $u(x)$ on D, i.e., the difference between $\operatorname{vrai}\max\limits_D u(x)$

and $\operatorname{vrai}\min_D u(x)$; $A_{k,\rho}$ is the set of points x in K_ρ for which $u(x) > k$; $u^{(k)}(x) = \max\{u(x) - k;\, 0\}$.

Definition. The boundary S of the domain D (or its part S_1) is said to satisfy condition (A) if there exist two positive numbers a_0 and θ_0 such that, for every ball K_ρ with center on S (respectively on S_1) of radius $\rho \leq a_0$ and for any component $\widetilde{D}_\rho$ of the intersection D_ρ of the ball K_ρ with D, the equality

$$\operatorname{mes} \widetilde{D}_\rho \leq (1 - \theta_0) \operatorname{mes} K_\rho.$$

Lemma 1. If $s/q_1 + (n-s)/q_2 > n-1$, $u(x) \in W_1^1(K_\rho)$, Ω is a measurable set belonging to K_ρ, $\Omega_0 = \{x \in K_\rho;\ u(x) \equiv 0\}$, then

$$\|u\|_{L(q_1,q_2)(\Omega)} \leq \beta \frac{\rho^{s/q_1 + (n-s)/q_2 + 1}}{\operatorname{mes}\Omega_0} \int_{K_\rho} |\operatorname{grad} u|\, dx;$$

β is a constant depending on n, s.

Corollary 1. If $s/q_1 + (n-s)/q_2 > n-1$, then for any function $u(x)$ from $W_1^1(K_\rho)$, for arbitrary k and l $(l \geq k)$, the estimate holds

$$(l-k)\|1\|_{L(q_1,q_2)(A_{l,\rho})} \le \beta \frac{\rho^{s/q_1+(n-s)/q_2+1}}{\operatorname{mes}(K_\rho \setminus A_{k,\rho})} \int_{A_{k,\rho}\setminus A_{l,\rho}} |\operatorname{grad} u|\, dx,$$

where β is the constant from Lemma 1.

Remark. For $q_1 = q_2 = 1$ we obtain the well-known result of De Giorgi (^{6}).

Let M, γ, δ be fixed numbers; let the numbers p_1, p_2 and $\varepsilon > 0$ satisfy the condition

$$s/p_1 + (n-s)/p_2 = n/2 - 1 + \varepsilon.$$

Definition. Denote by $B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ the set of functions $u(x)$ from $W_2^1(D)$ with $\operatorname{vrai\,max}_D |u| \le M$ and such that, for $\pm u(x)$ in any ball $K_\rho \subset D$ and for any $\sigma \in (0,1)$, the inequalities

$$\int_{A_{k,\rho(1-\sigma)}} |\operatorname{grad} u(x)|\, dx \le \gamma \left[(\sigma\rho)^{-2} \int_{A_{k,\rho}} |u^{(k)}|^2\, dx + \|1\|^2_{L(p_1,p_2)(A_{k,\rho})} \right] \tag{6}$$

hold for $k \ge \operatorname{vrai\,max}_{K_\rho} u(x) - \delta$.

Lemma 2. Let $u(x) \in B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ and $K_{\rho_0} \subset D$, where

$$\rho_0^\varepsilon < \operatorname*{vrai\,max}_{K_{\rho_0}} u(x) - k \equiv H < \delta;$$

then there exists a number θ such that from

$$\|1\|_{L(p_1,p_2)(A_{k,\rho_0})} \le \theta \rho_0^{n/2-1+\varepsilon}$$

it follows that

$$\operatorname*{vrai\,max}_{K_{\rho_0}/2} u(x) \le \tfrac{1}{2}\left(\operatorname*{vrai\,max}_{K_{\rho_0}} u(x) + k\right).$$

Lemma 3. There exists a positive number σ such that, for any ball K_ρ ($\rho < 1$) belonging to D together with the concentric ball $K_{4\rho}$, and for any function $u(x) \in B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$, at least one of the two inequalities holds:

$$\operatorname{osc}\{u, K_\rho\} \le 2^\sigma \rho^\varepsilon$$

or

$$\operatorname{osc}\{u, K_\rho\} \le (1 - 1/2^{\sigma-1}) \operatorname{osc}\{u, K_{4\rho}\}.$$

From Lemma 3 and Lemma 4.8 (^{2}), p. 90, it follows

Theorem 1. Let $u(x) \in B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ and $K_{\rho_0} \subset D$ ($\rho_0 \le 1$); then, for any ball K_ρ, $\rho \le \rho_0$, concentric with K_{ρ_0}, the estimate holds

$$\operatorname{osc}\{u, K_\rho\} \le c(\rho/\rho_0)^\alpha,$$

where $a = \min\{-\lg_4(1 - 1/2^{\sigma-1}),\ \varepsilon\}$, $c = 4^\alpha \max\{2M;\ 2^\sigma \rho_0^e\}$. The number σ is taken from Lemma 3.

To study the smoothness of functions from $B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ in domains adjacent to the boundary, we single out from $B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ a narrower class of functions satisfying inequalities of type (6) not only for interior balls, but also for balls intersecting the boundary S of the domain.

Definition. Denote by $B(D \cup S_1, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ the set of functions $u(x)$ from $B(D, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ which, together with $u(x)$, satisfy inequalities (6) also for balls K_ρ intersecting S_1, but not intersecting $S \setminus S_1$, for $k \ge \operatorname{vrai\,max}_{K_\rho \cap D} u(x) - \delta$ and $k \ge \operatorname{vrai\,max}_{K_\rho \cap S_1} u(x)$.

The domains of integration $A_{k,\rho}$ for balls intersecting S_1 in (6) are defined as the sets of points of $K_\rho \cap D$ at which $u(x) > k$.

Lemma 4. There exists a positive number σ such that for any ball K_ρ and the concentric ball $K_{4\rho}$ with center on S and $4\rho \le a/4$ ($a = \min(a_0; 1)$), and any function $u(x)$ from $B(\overline{D}, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ satisfying the condition

$$\operatorname{osc}\{u;\ S \cap K_\rho\} \le b\rho^e, \qquad e > 0,$$

at least one of the two inequalities holds:

$$\operatorname{osc}\{u;\ K_\rho \cap D\} \le 2^\sigma \rho^{\varepsilon_1}, \qquad \varepsilon_1 = \min\{\varepsilon, e\},$$

or

$$\operatorname{osc}\{u;\ K_\rho \cap D\} \le (1 - 1/2^{\sigma-1}) \operatorname{osc}\{u, K_{4\rho} \cap D\}.$$

Theorem 2. If S satisfies condition (A) and the function $u(x)$ from $B(D \cup S, M, \gamma, \delta, \varepsilon, s, p_1, p_2)$ satisfies on S the condition

$$\operatorname{osc}\{u;\ S \cap K_\rho\} \le b\rho^e, \qquad e > 0$$

for balls with centers on S and radius $\rho \le a_0$, then $u(x)$ satisfies a Hölder condition in $\overline{D}$, i.e. for any ball K_ρ

$$\operatorname{osc}\{u,\, K_\rho \cap D\} \le c\rho^\alpha.$$

The constants c and α are determined by known parameters.

On the basis of the results set forth and the results of work [2], the validity of the following theorem is established.

Theorem 3. Under conditions (2), (3), (4), the generalized solution $u(x)$ from $W_2^1(D)$ of equation (1) belongs to the class $C_{0,\alpha_1}(\overline{D'})$, where D' is an interior subdomain of the domain D.

If, moreover, $\operatorname{vrai\,max}_S u(x) < M$, then $u(x) \in C_{0,\alpha_2}(D)$.

If the boundary S satisfies condition (A) and $u|_S \in C_{0,\beta}(S)$, then $u(x) \in C_{0,\alpha_3}(\overline{D})$.

In all cases α_i and $|u|_{\alpha_i,D}$ are determined and estimated, respectively, by known quantities.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

Received
8 VII 1968

CITED LITERATURE

[1] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
[2] O. A. Ladyzhenskaya, N. N. Ural' tseva, *Linear and Quasilinear Equations of Elliptic Type*, "Nauka," 1964.
[3] O. L. Ladyzhenskaya, N. N. Ural' tseva, Izv. AN SSSR, Ser. Mat., part I, **26**, No. 1, 5 (1962); part II, **26**, No. 5, 753 (1962); part III, **27**, No. 1, 161 (1963).
[4] G. Stampacchia, *Problemi al contorno ellittici con dati discontinui dotati di soluzioni Holderiane*, Univ. Genova, 1960.
[5] C. M. Morrey, Math. Zs., **72**, 146 (1959).
[6] De Giorgi, Mem. Acc. Sci. Torino, Ser. 3, **3**, P. 1, 25 (1957).
[7] J. Nash, Am. J. Math., **80**, No. 4, 931 (1958).
[8] A. Kh. Gudiev, DAN, **180**, No. 6 (1968).
[9] A. Kh. Gudiev, DAN, **184**, No. 5 (1968).

Note: Figure translations are in progress. See original paper for figures.