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ALGEBRAS WITHOUT NILPOTENT ELEMENTS

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Abstract

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MATHEMATICS

Yu. M. RYABUKHIN

ALGEBRAS WITHOUT NILPOTENT ELEMENTS

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Below we shall consider algebras over a certain fixed associative-commutative ring Φ with identity. No additional restrictions on the ring Φ are assumed, unless the contrary is stated explicitly. The algebras under consideration are, as a rule, nonassociative. Associativity or alternativity of algebras is always stipulated separately. The identity element 1 of the ring Φ acts as the identity operator, i.e. $1x = x1 = x$ for every element x of an algebra R over Φ.

The aim of the present paper is to obtain structural theorems for algebras without nilpotent elements, i.e. for algebras in which the conditional identity

$$x^2 = 0 \Longleftrightarrow x = 0. \quad (1)$$

holds.

Algebras without nilpotent elements have been studied for a very long time. To obtain structural theorems, finiteness conditions were imposed, most often finite dimensionality. The first of the sufficiently general structural theorems was proved, apparently, by Weierstrass. In 1861 he characterized in his lectures finite-dimensional associative-commutative algebras without nilpotent elements over the field of real or complex numbers as finite direct sums of fields. In doing so, restrictions somewhat stronger than the absence of nilpotent elements were considered. By 1870 Dedekind had arrived at the same results, removing the superfluous restrictions.

From the results of Molin ([1]) and Cartan ([2]) it follows that in the Dedekind-Weierstrass theorem the requirement of commutativity may be omitted if one takes division rings instead of fields. And from the results of Wedderburn ([3]) it follows that instead of the field of complex numbers one may take an arbitrary field. Thus the following theorem is true:

Every finite-dimensional associative algebra over a field Φ, having no nilpotent elements, is a finite direct sum of division rings.

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A natural generalization of the notion of a field to the nonassociative case was the notion of an algebra with division. Algebras with left (right) division are those algebras R in which, for every nonzero element a and every $b \in R$, the equation $xa = b$ (respectively, $ax = b$) has a unique solution. Algebras with division are algebras with division both on the left and on the right. It is clear that every algebra with division (on the left and on the right) has no zero divisors and is a simple algebra, i.e. the only ideals in the algebra R with division are 0 and R.

It turns out that, in the theorem stated above, associativity cannot be omitted by replacing division rings by algebras with division. Namely, there exists a 4-dimensional special simple Jordan algebra over the field of real numbers, having no nilpotent elements, but having zero divisors and therefore not decomposable into a direct sum of algebras with division. On the other hand, the requirement of associativity can be weakened. From the results of Albert ([4]) and Zhevlakov ([5]) it follows that every finite-dimensional alternative algebra without nilpotent elements over an arbitrary field Φ (of characteristic $p \neq 2$) is a finite direct sum of algebras with division.

Naturally the question arises: what conditions must an algebra R over Φ satisfy in order that it be decomposable into a direct sum of division algebras?

To answer this question, let us consider the following requirements imposed on an algebra R over Φ:

(a) in the algebra R the conditional identity holds

$$x(yz) = 0 \Longleftrightarrow (xy)z = 0; \tag{2}$$

(b) the algebra R satisfies the minimality condition for principal ideals;

(c) for every element $a \in R$ the chain

$$R \supseteq Ra \supseteq (Ra)a \supseteq ((Ra)a)a \supseteq \ldots \tag{3}$$

stabilizes at a finite step;

(d) for every element $a \in R$ the chain

$$R \supseteq aR \supseteq a(aR) \supseteq a(a(aR)) \supseteq \ldots \tag{4}$$

stabilizes at a finite step.

It can be shown that the requirements (a)–(d) are independent. Everywhere below, by a direct sum we shall mean a ring-theoretic direct sum.

Theorem 1. *An algebra R over Φ decomposes into a direct (not necessarily finite) sum of left division algebras if and only if R has no nilpotent elements and the algebra R satisfies conditions (a)–(c). In order that the algebra R*

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decompose into a direct sum of division algebras, it is necessary and sufficient that R have no nilpotent elements and that conditions (a)–(d) hold.

Naturally the question arises about decomposition into a finite sum.

Theorem 2. *An algebra R over Φ decomposes into a finite direct sum of left division algebras if and only if R has no nilpotent elements, while in the algebra R conditions (a), (c) and the minimality condition for ideals hold. In order that an algebra R over Φ decompose into a finite direct sum of division algebras, it is necessary and sufficient that R have no nilpotent elements and that conditions (a), (c), (d) and the minimality condition for ideals hold.*

From Theorems 1 and 2 it obviously follows that

Theorem 3. *A finite-dimensional algebra R over a field Φ decomposes into a finite direct sum of division algebras if and only if R has no nilpotent elements and the identity (2) holds.*

The proof of Theorem 1 is based on the following lemmas:

Lemma 1. *If the algebra R is representable as a subdirect sum of algebras without zero divisors, then the relations (1) and (2) hold in R.*

Lemma 2. *If the conditional identities (1), (2) hold in the algebra R, then for every $a \in R$ the sets $\{x \in R \mid xa = 0\}$, $\{x \in R \mid ax = 0\}$ coincide and are ideals of the algebra R.*

Lemma 3. *If the relations (1), (2) hold in the algebra R and M is a minimal ideal in R, then the algebra M has no zero divisors.*

Lemma 4. *Let the conditional identities (1), (2) hold in R. Let $a \in R$, and let the chain (3) stabilize at some finite step. Then Ra is an ideal of the algebra R, $a \in Ra$, and $Ra = (Ra)a$.*

Lemma 5. *Let the algebra R over Φ decompose into a direct (not necessarily finite) sum of its ideals R_i, $R = \sum \oplus R_i$, $i \in \Delta$. If all the algebras R_i are simple, then R satisfies the minimality condition for principal ideals, and the R_i are precisely the minimal ideals of the algebra R. Moreover, for every nonzero ideal A in R there exists a nonempty $\Delta_A \subseteq \Delta$ such that $A = \sum \oplus R_i$, $i \in \Delta_A$. If all R_i are left division algebras, then every left ideal of the algebra R will be two-sided.*

It follows from Lemma 5 that the decompositions into a direct sum of division algebras considered in Theorems 1-3 are unique.

Recall that an algebra R is called **alternative** if every subalgebra of R generated by two generators is associative.

Lemma 6. *In every alternative algebra without nilpotent elements, the conditional identity (2) holds.*

From Theorem 3 and Lemma 6 it follows that

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Corollary 1. *A finite-dimensional alternative algebra R over a field Φ decomposes into a finite direct sum of division algebras if and only if R has no nilpotent elements.*

In the case of associative algebras we can prove a significantly stronger result, taking into account that associative division algebras on the left (on the right) are precisely fields.

Corollary 2. *An associative algebra R over Φ decomposes into a direct (finite direct) sum of fields if and only if R has no nilpotent elements, R satisfies the minimality condition for principal (respectively, for all) ideals, and for each element $a \in R$ the chain of principal left ideals*

$$(a)_l \supseteq (a^2)_l \supseteq (a^3)_l \supseteq \cdots$$

stabilizes after finitely many steps.

From this assertion follows the theorem of A. I. Gerchikov from [6].

Very often, in proving structural theorems analogous to Theorems 1-3, left (or right) Artinianness is used, i.e., the minimality condition for left (or right) ideals. This is the case, for example, for associative or alternative algebras. However, in the general case, in Theorems 1 and 2 one cannot replace requirements ( )–( ) by Artinianness. Namely, there exists an algebra R without zero divisors over an arbitrary field Φ, such that R has only finitely many one-sided ideals, the conditional identities (1), (2) hold in R, but the algebra R is not even a simple algebra. All the more, R will not be a division algebra on the left or on the right and, obviously, R does not decompose into a direct sum of such algebras.

Nevertheless, structural theorems can be proved even without any chain conditions. In this case, instead of direct sums, one has to consider subdirect sums.

Apparently, the first theorem of this kind was Krull' s theorem [7], stating that every associative-commutative algebra without nilpotent elements is a subdirect sum of algebras without zero divisors. In [8] it is shown that in Krull' s theorem the commutativity condition can be omitted, i.e., the following theorem is true:

Every associative algebra without nilpotent elements is a subdirect sum of algebras without zero divisors.

In this theorem the requirement of associativity cannot be omitted, as is shown by the already mentioned example of a 4-dimensional Jordan special simple algebra over the field of real numbers, having no nilpotent elements but having zero divisors.

The associativity requirement can be weakened. I. V. Lvov reported that (under certain restrictions on the characteristic) every alternative algebra without nilpotent elements is a subdirect sum of algebras without zero divisors. Below it will be shown that restrictions on the characteristic are superfluous. Namely, the following is true.

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Theorem 4. *An algebra R over Φ decomposes into a subdirect sum of algebras without zero divisors if and only if R has no nilpotent elements and satisfies the conditional identity (2).*

The proof of this theorem is based on Lemmas 1, 2 and the lemma:

Lemma 7. *Let the conditional identities (1), (2) hold in an algebra R, and let $\mathfrak{M}$ be a subgroupoid of the multiplicative groupoid $R(\cdot)$ of the algebra R. Let P be the set of all such $x \in R$ that $xy = 0$ for some $y \in \mathfrak{M}$. Then P is an ideal of the algebra R. If $\mathfrak{M}$ is a maximal subgroupoid in R not containing zero 0, then $P = R \setminus \mathfrak{M}$. The subgroupoid $\mathfrak{N}_x$ of the groupoid*

$R(\cdot)$, generated by the nonzero element $x \in R$, does not contain zero 0.

From Lemma 6 and Theorem 5 it follows

Corollary 3. *An alternative algebra R over Φ decomposes into a direct sum of algebras without zero divisors if and only if R has no nilpotent elements.*

Let us now note that Theorem 4 easily implies

Theorem 5. *An algebra R decomposes into a subdirect sum of algebras without zero divisors if and only if every subalgebra in R generated by three (not necessarily distinct) generators decomposes into a subdirect sum of algebras without zero divisors.*

There exists a Jordan algebra with three generators such that every subalgebra generated by two generators decomposes into a subdirect sum of algebras without zero divisors, although the whole algebra does not decompose into a subdirect sum of algebras without zero divisors.

Institute of Mathematics with Computing Center
Academy of Sciences of the Moldavian SSR

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