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MATHEMATICS

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HOMOGENEOUS SUBSETS OF ABSOLUTES

(Presented by Academician P. S. Aleksandrov on 25 IV 1969)

I. The proof, given by B. A. Efimov in the paper (³), of the existence of a broad class of essentially different homogeneous extremally disconnected spaces, obtained without the use of any set-theoretic hypotheses, naturally leads to the consideration of extremally disconnected topological groups. In the paper (¹) A. V. Arkhangel' skii proves some assertions about them, and in the paper (⁸) (under the assumption of the continuum hypothesis) we prove the existence of a topological countable group whose space is extremally disconnected.

Let $\mathfrak{x} = E\{U_\alpha \mid \alpha \in A\}$ be some ultrafilter of open subsets of the space X; let G be a group of homeomorphisms of X onto itself. To each $g \in G$ we associate the ultrafilter $\mathfrak{x}_g = E\{g(U_\alpha) \mid \alpha \in A\}$, define the set $\mathrm{Orb}(\mathfrak{x}, G) = E\{\mathfrak{x}_g \mid g \in G\}$, considered below as a subspace of pX, and define on it the action of the group G in the following way: $h\mathfrak{x}_g = \mathfrak{x}_{gh}$.

In the case when the topological group G is considered as the natural group of its homeomorphisms, i.e. $X = G$, and the ultrafilter concerns the identity $e_G \in G$, we shall denote $\mathrm{Orb}(\mathfrak{x}, G)$ by $G^*(X)$. In this case, in view of the relative simplicity of the construction and the absence of set-theoretic hypotheses, it is natural to consider the question (B. A. Efimov):

Problem A. In what case is the space $G^*(\mathfrak{x})$, with the operation $*$ induced by the group G: $\mathfrak{x}_g * \mathfrak{x}_h = \mathfrak{x}_{gh}$, a topological group?

Of course, if the space of the group G is extremally disconnected, then the ultrafilter $\mathfrak{x}$ is unique, and the answer is therefore positive. In the present paper we prove the impossibility of constructing new extremally disconnected groups by the method described, and also derive some properties of homogeneous subsets of absolutes.

II. Theorem 1. *If G is a group of homeomorphisms of a topological space X onto itself, then G is also a group of homeomorphisms of the space* $\mathrm{Orb}(\mathfrak{x}, G)$ *onto itself, for any choice of $\mathfrak{x} \in pX$.*

Proof. First of all, the following chain of equalities

$$h\,\mathrm{Orb}(\mathfrak{x},G)=hE\{\mathfrak{x}_g\mid g\in G\}=E\{h\mathfrak{x}_g\mid g\in G\}=E\{\mathfrak{x}_{gh}\mid g\in G\}=$$

$$=E\{\mathfrak{x}_g\mid g\in Gh^{-1}\}=E\{\mathfrak{x}_g\mid g\in G\}=\mathrm{Orb}(\mathfrak{x},G)$$

shows that h is a mapping onto. Recall that a basic neighborhood of an ultrafilter $\mathfrak{F}$ as a point of the absolute is uniquely determined by each element $U\in\mathfrak{F}$ as the set of ultrafilters containing U. Then let W^{α}_{gh} be a neighborhood of the ultrafilter $\mathfrak{x}_{gh}=h\mathfrak{x}_g$, given by the open set $ghU_\alpha\in\mathfrak{x}_{gh}$. But $gU_\alpha\in\mathfrak{x}_g$, and, as is not difficult to verify, $W^{\alpha}_{gh}=hW^{\alpha}_{g}$, where W^{α}_{g} is a neighborhood of the ultrafilter $\mathfrak{x}_g$, given by the set gU_α. Since the preceding argument is also true for h^{-1}, h is a homeomorphism, and, by the arbitrariness of the choice of $h\in G$, it follows that G is a group of homeomorphisms of the space $\mathrm{Orb}(\mathfrak{x},G)$. The theorem is proved.

Theorem 2. *Let X be a topological space, G a group of its*

homeomorphisms such that $X=[G(x_0)]$ for some $x_0\in X$, and $\mathfrak{X}=E\{U_\alpha\mid\alpha\in A\}$ is a certain ultrafilter of open subsets of the space X touching the point x_0, i.e. $x_0\in\bigcap\{[U_\alpha]\mid\alpha\in A\}$. Then $\mathrm{Orb}(\mathfrak{X},G)$ is an everywhere dense homogeneous extremally disconnected subspace of pX.

From Theorem 2 there follows the following generalization of Theorem 6 of [3]:

Corollary 1. The absolute pX of any homogeneous topological space X decomposes into everywhere dense disjoint topologically homogeneous absolutes on which one and the same group of homeomorphisms acts, this group being a group of homeomorphisms of the space X.

Corollary 2. If in a space X every class of homogeneity is everywhere dense, then its absolute decomposes into everywhere dense homogeneous disjoint absolutes.

III. Let us now turn to the consideration of Problem A.

Theorem 3. If Y is a topological space, X is its extremally disconnected subspace, and for every $y\in Y$ there exists a homeomorphism φ_y of the space Y onto itself such that $\varphi_y(y)\in X\cap\mathrm{int}[X]$, then the space Y is extremally disconnected.

Proof. Let U_1 and U_2 be an arbitrary pair of disjoint open subsets of the space Y. To prove the theorem it is, obviously, sufficient for us to show that $[U_1]\cap[U_2]=\emptyset$. Let $y\in[U_1]$ for some $y\in Y$. By the hypothesis, taking into account that φ_y is a homeomorphism, we have $\varphi_y(y)\in X\cap\mathrm{int}[X]\cap[\varphi_y(U_1)]$, but $[\varphi_y(U_i)]_Y\cap X\cap\mathrm{int}[X]=[\varphi_y(U_i)]_X\cap X\cap\mathrm{int}[X]$, $i=1,2$, and then from the extremal disconnectedness of X it follows that

$$X\cap[\varphi_y(U_1)]_Y\cap[\varphi_y(U_2)]_Y\cap\mathrm{int}[X]=\emptyset,$$

and hence $\varphi_y(y) \notin [\varphi_y(U_2)]$, i.e. $y \notin [U_2]$. In view of the arbitrariness of the choice of $y \in [U_1]$, it follows that $[U_1] \cap [U_2] \neq \emptyset$. The theorem is proved.

Corollary 3. If X is an extremally disconnected subspace of a homogeneous topological space Y and $\operatorname{int}[X] \neq \emptyset$, then Y is also extremally disconnected.

Corollary 4. If the space of a subgroup H of a topological group G is extremally disconnected, then at least one of the following two assertions is true:

1°. The space of the group G is extremally disconnected.

2°. $\operatorname{int}[H] = \emptyset$.

Theorem 4. If G and H are topological groups, with H topologically embedded in the absolute pG of the group G in such a way that the restriction $\varphi|H$ of the canonical mapping $\varphi : pG \to G$ is a homomorphism of the group H, then at least one of the following two assertions is true:

1°. The space of the group G is extremally disconnected and $\varphi|H$ is an isomorphism.

2°. $\operatorname{int}[H]_{pG} = \emptyset$.

We shall precede the proof of the theorem by the following two lemmas.

Lemma 1. Every semi-open homomorphism of a topological group is open.

Proof. Let G and H be topological groups and $\varphi : H \to G$ a semi-open homomorphism; for the proof it is sufficient for us to show that if U is a neighborhood of the identity e_h of the group H, then $e_g \in \operatorname{int}\varphi(U)$, where e_g is the identity element of the group G. Choose a neighborhood V of the point e_h such that $V = V^{-1}$, $V^2 \subset U$; this can always be done according to the axiomatics of a topological group (see, for example, [6]). Since φ is a homomorphism, we have

$$\varphi(V)(\varphi(V))^{-1} = \varphi(V)\varphi(V^{-1}) = \varphi(VV^{-1}) = \varphi(V^2) \subset \varphi(U)$$

and, by the definition of a semi-open mapping, $\operatorname{int}\varphi(V) \neq \emptyset$; since $e_g \in WW^{-1}$ for every open subset W of the group G, it follows that

$$e_g \in (\operatorname{int}\varphi(V))(\operatorname{int}\varphi(V))^{-1} \subset \varphi(V)\varphi(V^{-1}),$$

and hence $e_g \in \operatorname{int}\varphi(U)$, as required. The lemma is proved.

Lemma 2. Every irreducible homomorphism of a topological group is a monomorphism.

Proof. Let $h' \in H \cap \varphi^{-1}(e_g)$. Then, since φ is a homomorphism, $\varphi(hh') = \varphi(h)$ for every $h \in H$. But if $h' \neq e_h$, then for some neighborhood $U \subset H$ of the point e_h we have $U \cap (Uh') = \emptyset$, $\varphi(U) = \varphi(Uh')$, and

$$\varphi^{\#}(U) = \varphi(U) \setminus \varphi(H \setminus U) \subset \varphi(U) \setminus \varphi(Uh') = \emptyset,$$

which contradicts the irreducibility of the mapping φ. Consequently, $h' = e_h$ for every $h' \in \varphi^{-1}(e_g)$, i.e. $\varphi^{-1}(e_g) = \{e_h\}$, whence it follows that φ is a monomorphism. The lemma is proved.

Corollary 5. *Every irreducible semi-open homomorphism of a topological group is an isomorphism.*

Proof of Theorem 4. Suppose that 2^0 does not occur, i.e. $\operatorname{int}(H) \neq \emptyset$. Since the mapping φ is irreducible and, under irreducible mappings, (5) the inverse image of a nowhere dense set is nowhere dense, it follows from our assumption that

$$\operatorname{int}[\varphi(H)] \neq \emptyset.$$

But, by the condition of the theorem, $\varphi|H$ is a homomorphism; consequently, $G_1 = \varphi(H)$, and hence also $G_2 = [\varphi(H)]$, are subgroups of the group G. From the fact that $\operatorname{int} G_2 \neq \emptyset$ it follows that $G_2 = \operatorname{int} G_2$, i.e. G_2 is open, and hence an open-closed subgroup of the group G, i.e. an open-closed subset of its space; and one may assume that $pG_2 = \varphi^{-1}(G_2)$ and that $\varphi|\varphi^{-1}(G_2)$ is the canonical mapping of pG_2 onto G_2. Since $G_1 = \varphi(H)$ is dense in G_2, and the mapping $\varphi|\varphi^{-1}(G_2)$ is irreducible, it follows that $H = \varphi^{-1}(G_1)$ is dense in pG_2, and therefore is extremally disconnected as a dense subset of an extremally disconnected space. It is known that the restriction of the canonical mapping of an absolute to a dense subspace is irreducible and semi-open; consequently, $\varphi|H$ is an irreducible semi-open homomorphism, and by Corollary 5 $\varphi|H$ is an isomorphism, and $G_1 = \varphi(H)$ is also an extremally disconnected group. But $G_2 = [G_1]$, and by Corollary 2 the space of the group G_2 is also extremally disconnected. As we noted above, G_2 is an open-closed subgroup of the group G, i.e. the space of the group G is homeomorphic to the free sum of spaces homeomorphic to G_2, whence the extremal disconnectedness of the space of the group G follows. Thus we have shown that from $\operatorname{int}[H] \neq \emptyset$ it follows that the group G is extremally disconnected and $\varphi|H$ is an isomorphism, which proves the theorem completely.

Let us note that Theorem 4 contains an exhaustive solution of problem (A).

IV. Let us consider in somewhat greater detail the space $G^*(\mathfrak{x})$. In the paper (1) A. V. Arhangel' skii showed that every bicompact subset of a topological group whose space is extremally disconnected is finite. From Theorem 4 it follows that the homogeneous space $G^*(\mathfrak{x})$ with the natural operation is not, in general, a topological group, but nevertheless an analogous assertion is also true for it; moreover, the method used in (1) goes through almost without changes in this case as well.

Remark 1. Let G be a semitopological group, i.e. the group operation is continuous in each variable, and let H be some subgroup of G. Exactly as in the case of topological groups, one can define the space of adjacent classes G/H (see, for example, (6)), and the natural mapping of G onto G/H is again continuous and open. The difference consists only in the fact that we cannot guarantee any separation properties (except T_1, if $[H] = H$).

Theorem 5. *If G is a topological group and $\mathfrak{x}$ is an arbitrary ultrafilter of open subsets of G touching the identity e_g of the group, then every bicompact subset of the space $G^*(\mathfrak{x})$ is finite.*

Proof. As follows from Theorem 1, every element of the group G, considered as an action on $G^*(\mathfrak{x})$, is a homeomorphism. Considering the operation $\mathfrak{x}_g * \mathfrak{x}_h = \mathfrak{x}_{gh}$, where g and h are in G, we note that

$$\mathfrak{x}_g * \mathfrak{x}_h = h\mathfrak{x}_g = h_{g^{-1}}\mathfrak{x}_h,$$

whence it follows that $G^*(\mathfrak{x}) * \mathfrak{x}_h$ and $\mathfrak{x} * D^*(\mathfrak{x})$ are homeomorphisms of the space $G^*(\mathfrak{x})$, i.e. G^* is a semitopological group. Suppose now that in $G^*(\mathfrak{x})$ there exists an infinite bicom-

closed subset. Then, as is not hard to see, by virtue of the homogeneity of $G^*(\mathfrak{X})$, for some sequence $\mathfrak{N} = E\{\mathfrak{x}_{g_n} \mid n \in N\}$ one may assume that $\mathfrak{x} \in [\mathfrak{N}] \setminus \mathfrak{N}$, and the set $[\mathfrak{N}]$ is bicompact. Owing to the continuity of the canonical mapping $\varphi : pG \to G$, and by the construction, $e_g \notin \varphi(\mathfrak{N}) = E\{g_n \mid n \in N\}$, and $[\varphi(\mathfrak{N})]$ is a bicompact set containing e_g. But, as indicated in (1), one can choose in G a closed subgroup H of type G_δ such that $H \cap E\{g_n \mid n \in N\} = \emptyset$. Since $\varphi|G^*(\mathfrak{X})$ is a homeomorphism, $H^* = G^*(\mathfrak{X}) \cap \varphi^{-1}(H)$ is a subgroup of $G^*(\mathfrak{X})$, and, moreover, $H^* * \mathfrak{x}_g = \varphi^{-1}(Hg) \cap G^*(\mathfrak{X})$. Further, let $g \in G$ and let U be a neighborhood of e_g in G such that $HU \cap HgU = \emptyset$. Then, obviously, $U^* = \varphi^{-1}(U) \cap G^*(\mathfrak{X})$ is a neighborhood (by the continuity of φ) of the point $\mathfrak{x}$ in $G^*(\mathfrak{X})$, and again

$$H^* * U^* \cap H^* * \mathfrak{x}_g * U^* = \emptyset,$$

whence, owing to the arbitrariness of the choice of $g \in G$, the Hausdorffness of the space $G^*(\mathfrak{X})/H^*$ follows; i.e., taking Remark 1 into account, one may conclude that $G^*(\mathfrak{X})/H^*$ is a Hausdorff extremally disconnected space, as an open continuous image of the extremally disconnected space $G^*(\mathfrak{X})$ (^{1}). But H^* is a closed subset of type G_δ in $G^*(\mathfrak{X})$ as the inverse image of the closed G_δ under a continuous mapping, $\mathfrak{x} \in H^* \cap ([\mathfrak{N}] \setminus H^*) \neq \emptyset$, and, consequently, H^* is a nonisolated point with the first axiom of countability in the image of the bicompactum $[\mathfrak{N}]$, i.e. a χ-point (^{1,7}), which cannot occur in the Hausdorff extremally disconnected space $G^*(\mathfrak{X})/H^*$ (^{7}). It follows from this that our assumption on the existence of an infinite bicompact subset in $G^*(\mathfrak{X})$ is false. The theorem is proved.

V. The following is also connected with the solution of problem ($*$).

Theorem 6. *The space of a nondiscrete topological group cannot be a dense subset of the absolute of a separable metrizable space.*

Lemma 3. *The π-weight of the space of a topological group is equal to its weight.*

Proof. In the case when the group is finite the assertion of the lemma is obvious; therefore let G be an infinite group and let $\mathfrak{B} = E\{V_\beta \mid \beta \in B\}$ be a π-base of its space. If U is an arbitrary neighborhood of the identity $e_g \in G$, then there exists a neighborhood $V \ni e_g$ such that $VV^{-1} \subset U$; by the property of a π-base, $V_\beta \subset V$ for some $\beta \in B$, and $e_g \in V_\beta V_\beta^{-1} \subset VV^{-1} \subset U$, whence it follows that the family

$$\mathfrak{B}\mathfrak{B}^{-1} = E\{V_\alpha V_\beta^{-1} \mid (\alpha, \beta) \in B^2\}$$

is a base at the point e_g. From this, as is not hard to verify, it follows that the family

$$\mathfrak{B}\mathfrak{B}^{-1}\mathfrak{B} = E\{V_\alpha V_\beta^{-1} V_\gamma \mid (\alpha, \beta, \gamma) \in B^3\}$$

is a base for the space of the group G; owing to the infinitude of G we have $\pi\omega G \geq \aleph_0$, and $\omega G = (\pi\omega G)^3 = \pi\omega G$, as was required. The lemma is proved.

Proof of Theorem 6. It is known that π-weight does not change when passing to dense subsets. Then, if the space of a nondiscrete topological group is topologically embedded as a dense subset in the absolute pX of a space X with a countable base, then $\pi\omega pX = \pi\omega G$. From the countable weight of X follows the countable π-weight of pX, and, taking account of the preceding equality and Lemma 3, the countable weight of G follows. At the same time G is extremally disconnected as a dense subset of the absolute, is nondiscrete by assumption, and its weight, evidently, is uncountable. Consequently, our assumption on the embedding of G in pX is false. The theorem is proved.

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REFERENCES

[1] A. Arhangel' skii, C. R., **265**, 822 (1967).
[2] A. Gleason, Ill. J. Math., **2**, No. 4, 482 (1958).
[3] B. Efimov, DAN, **179**, No. 2, 271 (1968).
[4] S. Iliadis, S. Fomin, UMN, **21**, No. 4, 47 (1966).
[5] V. I. Ponomarev, UMN, **21**, issue 4, 101 (1966).
[6] L. S. Pontryagin, *Continuous Groups*, Moscow, 1954.
[7] S. Sirota, DAN, **178**, No. 3, 542 (1968).
[8] S. Sirota, Matem. sborn., **79**(121), No. 2 (6) (1969).

Note: Figure translations are in progress. See original paper for figures.