NEW INTEGRAL TRANSFORMS

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MATHEMATICS

M. L. RASULOV, I. S. ZEYNALOV

NEW INTEGRAL TRANSFORMS

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In the book [1], by the method of contour integration, the mixed problem was solved:

$$c(x)M\left(t, \frac{\partial}{\partial t}\right)V = L\left(x, \frac{\partial}{\partial x}\right)V + f(x,t); \tag{1}$$

$$\lim_{x\to y} B\left(y, \frac{d}{dn_y}, \frac{\partial}{\partial t}\right)V(x,t) = \psi(y,t), \qquad y \in \mathcal{T}; \tag{2}$$

$$V(x,0) = \Phi(x) \tag{3}$$

in the three-dimensional domain D with Lyapunov-type boundary $\mathcal{T}$, in the case when the dependence on t of the free term $\psi(y,t)$ of the boundary condition (2) has the special form:

$$\psi(y,t) = \psi(y)\exp B_1(t), \tag{4}$$

where

$$M\left(t, \frac{\partial}{\partial t}\right) = b_0(t)\frac{\partial}{\partial t} + b_1(t), \qquad B_0(t) = \int_0^t b_0^{-1}(\tau)\,d\tau,$$

$$B_1(t) = \int_0^t b_0^{-1}(\tau)b_1(\tau)\,d\tau, \qquad L\left(x, \frac{\partial}{\partial x}\right) = \sum_{i=1}^{3}\left(\frac{\partial^2}{\partial x_i^2} + a_i(x)\frac{\partial}{\partial x_i}\right) + a(x),$$

$$B\left(y, \frac{d}{dn_y}, \frac{\partial}{\partial t}\right) = \left[\alpha_1(y) + \alpha_2(y)M\left(t, \frac{\partial}{\partial t}\right)\right]\frac{d}{dn_y} + \alpha_2(y)\alpha_3(y)M\left(t, \frac{\partial}{\partial t}\right) +$$

$$+\alpha_4(y),$$

$b_0(t) > 0$, $b_1(t)$ are continuous functions on the interval $[0, \infty)$; the functions $c(x)$, $a_i(x)$, $a(x)$ are continuously differentiable in the domain $D + \mathcal{T}$, and if D is an unbounded domain (in the case of an exterior problem), then all these functions are assumed bounded in $D+\mathcal{T}$; the function $c(x)$ is positive in $D+\mathcal{T}$; $\alpha_i(y)$, $\psi(y)$ are continuous on $\mathcal{T}$.

Let R be a sufficiently large and δ a sufficiently small positive number. Denote by R_δ the domain of complex values λ satisfying the inequalities:

$$|\lambda| \geq R, \qquad \cos\arg\lambda \geq \delta.$$

Denote by S an infinite open contour situated in the domain R_δ, with the part of the contour S lying outside a circle of sufficiently large radius with center at the origin of the λ-plane coinciding with the continuations of the rays

$$\cos\arg\lambda = \delta.$$

It has been proved [1] that problem (1)–(3), for smooth $\Phi(x)$, $f(x,t)$, has a solution $V(x,t)$ of the form

$$V(x,t) = V_1(x,t) + V_2(x,t),$$

where

$$V_1(x,t) = \frac{1}{\pi\sqrt{-1}} \int_S \exp\left[\lambda^2 B_0(t) + B_1(t)\right] \frac{u_1(x,\lambda)}{\lambda}\, d\lambda; \tag{5}$$

$$V_2(x,t) = \frac{1}{\pi\sqrt{-1}} \int_S \lambda\, d\lambda \int_D G(x;\xi,\lambda) y(t,\xi,\lambda)\, dD_\xi; \tag{6}$$

$u_1(x,\lambda)$ is the solution of the spectral problem

$$L\left(x, \frac{\partial}{\partial x}\right) u - \lambda^2 c(x) u = \Phi(x), \tag{7}$$

$$\lim_{x\to y} B\left(y, \frac{d}{dn_y}, \lambda^2\right) u(x,\lambda) = \psi(y) \tag{8}$$

for the corresponding homogeneous equation (7); $G(x,\xi,\lambda)$ is the Green' s function of problem (7)–(8); $y(t,\xi,\lambda)$ is the solution of the Cauchy problem

$$M(t, \partial/\partial t)y - \lambda^2 y = f(\xi, t), \qquad y(0) = c(\xi)\Phi(\xi),$$

where $V_1(x,t)$ is the solution of problem (1)–(3) for $\Phi(x) \equiv 0$, $f(x,t) \equiv 0$; $V_2(x,t)$ is the solution of problem (1)–(3) for $\psi(y,t) \equiv 0$.

In order that problem (1)–(3) could be solved by the method of contour integrals for an arbitrary sufficiently smooth function $\psi(y,t)$, first of all one must associate with the mixed problem (1)–(3) a suitable boundary-value problem with parameter λ, in which the free term of the boundary condition will be a representation of $\psi(y,t)$ in the form of the contour integral (5). This idea has led to new mutually inverse integral transformations, to which the present note is devoted.

Now suppose that $b_0(t)$, $b_1(t)$ are continuous; moreover, $b_1(t)$ is bounded on the interval $[0, +\infty)$, and for $b_0^{-1}(t)$ the condition

$$0 < \varepsilon \le b_0^{-1}(t) \le Ce^{\sigma t}, \tag{9}$$

is satisfied, where $C > 0$, $\sigma \ge 0$, ε are constant numbers.

Let $f(t)$ be continuous together with its derivative on each finite part of the interval $[0, +\infty)$, except for a finite number of points at which they may have discontinuities of the first kind. Suppose, moreover, that

$$|f(t)| \le C_1 e^{\sigma_1 t}, \tag{10}$$

where $C_1 > 0$, $\sigma_1 \ge 0$ are constants.

Denote by $\Pi(R, a)$ the domain of values of λ satisfying the inequalities:

$$\operatorname{Re}\lambda^2 \ge a, \qquad |\lambda| \ge R. \tag{11}$$

Definition. The function $F_1(\lambda)$, defined by the formula

$$F_1(\lambda) = \int_0^\infty \exp[-\lambda^2 B_0(t) + B_1(t)] B_0'(t) f(t)\, dt, \tag{12}$$

will be called the **integral transform** of the function $f(t)$ with respect to the differential operator $M(t, \partial/\partial t)$.

We note that $F_1(\lambda)$ is an analytic function in the domain $\Pi(R, a)$. Indeed, we have

$$\left|\int_0^\infty \exp[-\lambda^2 B_0(t) + B_1(t)] B_0'(t) f(t)\, dt\right| \le$$

$$\leq \int_0^\infty \exp[-B_0(t)\operatorname{Re}\lambda^2 + c_1 B_0(t)]B_0'(t)|f(t)|\,dt,$$

where c_1 is a constant bounding $|b_1(t)|$ from above.

From the last inequality, in accordance with (9), for $a > c_1$ we have

$$\left|\int_0^\infty \exp[-\lambda^2 B_0(t) + B_1(t)]B_0'(t)f(t)\,dt\right| \leq \frac{CC_1}{a\varepsilon - c_1\varepsilon - \sigma - \sigma_1}.$$

In exactly the same way one can prove the uniform convergence of the integral obtained from (12) by formal differentiation with respect to λ in the domain $\Pi(R,a)$.

From the last inequality it is seen that, under the condition

$$a\varepsilon > c_1\varepsilon + \sigma_1 + 2\sigma$$

the integral (12) and the integral obtained from it by differentiation with respect to λ converge uniformly for all $\lambda \in \Pi(R,a)$, and they define analytic functions of λ.

Definition. A complex function $f(t)$, defined on the interval $[0,\infty)$, will be called an **original with respect to the operator** $M(t,\partial/\partial t)$, if the following conditions are satisfied:

1) The function $f(t)$ is continuous together with its first derivative on every finite part of the interval $[0,+\infty)$, with the exception of a finite number of points at which they may have discontinuities of the first kind.
2) Condition (10) is satisfied.
3) The function $F_1(\lambda)$, defined by the integral (12), is analytically continuable to the whole domain R_δ, and its analytic continuation $F(\lambda)$ in the domain R_δ tends uniformly with respect to $\arg\lambda$ to zero as $|\lambda| \to +\infty$.

Theorem. If the function $f(t)$ is an original with respect to the operator $M(t,\partial/\partial t)$, then at every point of continuity it is representable in the form of the contour integral

$$f(t) = \frac{1}{\pi\sqrt{-1}}\int_S \exp[\lambda^2 B_0(t) - B_1(t)]\lambda F(\lambda)\,d\lambda. \tag{13}$$

Proof. Let r_n be a sequence of positive numbers such that $\lim_{n\to\infty} r_n = +\infty$. Denote by O_n the circle of radius r_n with center at the origin of the λ-plane.

Let N_n, L_n, E_n, D_n be, respectively, the points of intersection of the circle O_n with the contour S, with the bisector of the first coordinate angle, with the curve

$$\operatorname{Re}\lambda^2 = a \tag{14}$$

and with the real axis of the λ-plane. Let further N'_n, L'_n, E'_n be the points symmetric to the points N_n, L_n, E_n with respect to the real axis.

Obviously, according to Cauchy' s theorem for analytic functions, we have

$$\int_S \exp[\lambda^2 B_0(t) - B_1(t)]\lambda F(\lambda)\, d\lambda = \lim_{n\to\infty} \int_{N'_n\widehat{D_{nN}}n} \exp[\lambda^2 B_0(t) - B_1(t)]\lambda F(\lambda)\, d\lambda =$$

$$= \lim_{n\to\infty} \left\{ \int_{\widehat{N'_n L_n}} + \int_{\widehat{L'_n E_n}} + \int_{\widehat{E_{nE}}n} + \int_{\widehat{E'_n L_n}} + \int_{\widehat{L_{nN}}n} \right\} \exp[\lambda^2 B_0(t) - B_1(t)]\lambda F(\lambda)\, d\lambda. \tag{15}$$

Denote by $C(r_n)$ the maximum of $|F(\lambda)|$ on the arc $N'_n\widehat{D_{nN}}n$ of the circle O_n. According to the condition of the theorem,

$$\lim_{n\to\infty} C(r_n) = 0, \tag{16}$$

and the convergence is uniform with respect to $\arg\lambda$ in the domain R_δ.

Integrating by parts, we obtain

$$\left| \int_{\widehat{N'_n L_n}} \exp[\lambda^2 B_0(t) - B_1(t)]\lambda F(\lambda)\, d\lambda \right| \le \exp[-\operatorname{Re} B_1(t)] C(r_n) \times$$

$$\times \int_{-\pi/2+\gamma}^{-\pi/4} \exp[r_n^2 B_0(t)\cos 2\theta]\, r_n^2\, d\theta = \exp[-\operatorname{Re} B_1(t)] C(r_n) \times$$

$$\times \left\{ - \frac{\exp[r_n^2 B_0(t)\cos 2\theta]}{2B_0(t)\sin 2\theta} \Bigg|_{-\pi/2+\gamma}^{-\pi/4} - \frac{1}{B_0(t)} \int_{-\pi/2+\gamma}^{-\pi/4} \frac{\exp[r_n^2 B_0(t)\cos 2\theta]}{\sin^2 2\theta \cos^{-1} 2\theta}\, d\theta \right\},$$

where $-\pi/2 + \gamma = -\arccos\delta$.

In view of the boundedness of the sum enclosed in braces on the right-hand side of the last inequality, on the basis of (16) we conclude that

$$\lim_{n\to\infty}\int_{\widehat{N_{nL}n}} \exp[\lambda^2 B_0(t) - B_1(t)]\,\lambda F(\lambda)\,d\lambda = 0. \tag{17}$$

It is shown in exactly the same way that the limit of the last integral in braces in (15) is equal to zero.

Let us estimate the second integral in (15):

$$\left|\int_{\widehat{L_{nE}n}} \exp\{[\lambda^2 B_0(t) - B_1(t)]\lambda F(\lambda)\}\,d\lambda\right| \le C(r_n)\exp[-\operatorname{Re} B_1(t)]\times$$

$$\times\int_{-\pi/4}^{-\frac{1}{2}\arccos ar_n^{-2}} \exp[r_n^2 B_0(t)\cos 2\theta]\,r_n^2 d\theta \le$$

$$\le C(r_n)\exp[-\operatorname{Re} B_1(t)]\exp[aB_0(t)]\,r_n^2\left[-\frac{1}{2}\arccos ar_n^{-2} + \frac{\pi}{4}\right].$$

In view of the uniform boundedness of
$r_n^2[-\frac{1}{2}\arccos ar_n^{-2} + \pi/4]$, from the last inequality we conclude that

$$\lim_{n\to\infty}\int_{\widehat{L_{nE}n}} \exp[\lambda^2 B_0(t) - B_1(t)]\,\lambda F(\lambda)\,d\lambda = 0. \tag{18}$$

In exactly the same way the limit of the penultimate integral in braces in (15) is equal to zero. As for the third integral in (15), for it, according to (12), we have

$$\int_{\widehat{E_{nE}n}} \exp[\lambda^2 B_0(t) - B_1(t)]\,\lambda F(\lambda)\,d\lambda = \int_{\widehat{E_{nE}n}} \exp[\lambda^2 B_0(t) - B_1(t)]\,\lambda\,d\lambda\times$$

$$\times\int_0^\infty \exp[-\lambda^2 B_0'(\tau) + B_1(\tau)]B_0'(\tau)f(\tau)\,d\tau = \int_0^\infty \exp[B_1(\tau)-$$

$$- B_1(t)]B_0'(\tau)f(\tau)\,d\tau\int_{\widehat{E_{nE}n}} \exp[\lambda^2(B_0(t) - B_0(\tau))]\,\lambda\,d\lambda = \int_0^\infty \exp\{a[B_0(t)-$$

$$- B_0(\tau)] + [B_1(\tau) - B_1(t)]\}B_0'(\tau)f(\tau)\,\frac{\sin\sqrt{r_n^4 - a^2}\,[B_0(t) - B_0(\tau)]}{B_0(t) - B_0(\tau)}\,d\tau.$$

Consequently, by the known formula from (15), taking into account (18), (19), and the remarks to these equalities, we obtain formula (13), which was required to prove.

From this theorem follows the validity of the results formulated above from the book [1], relating to the solution of problem (1)–(3) in the general case when $\psi(y, t)$ is an original in the sense of the present note.

Azerbaijan State University
named after S. M. Kirov
Baku

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REFERENCES

1. M. L. Rasulov, *The Method of the Contour Integral*, "Nauka," 1964.

Note: Figure translations are in progress. See original paper for figures.