ON THE REPRESENTATION OF SOLVABLE GROUPS BY MATRICES OVER A CERTAIN FIELD OF CHARACTERISTIC ZERO

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Abstract

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ON THE REPRESENTATION OF SOLVABLE GROUPS BY MATRICES OVER A CERTAIN FIELD OF CHARACTERISTIC ZERO

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One of the basic results in the theory of finite-dimensional representations of solvable groups is the theorem of A. I. Mal' tsev ([1]): a solvable matrix group has a normal divisor of finite index whose commutant is nilpotent. In connection with this theorem the question arises: what abstract properties must a solvable group possess in order that it can be represented isomorphically by matrices over some field? As an example one may cite the following assertion: every polycyclic group is representable by matrices over the field of rational numbers ([2], [3]).

We give the definitions needed in what follows. Let G be an arbitrary locally nilpotent torsion-free group; Ω an arbitrary field of characteristic zero, and x^λ a single-valued function which assigns to any elements $x \in G$ and $\lambda \in \Omega$ a certain element of G.

G is called an Ω-powered group ([4]) if the following conditions are satisfied:

1) $x^1 = x, \quad x^{\lambda+\mu} = x^\lambda x^\mu, \quad x^{\lambda\mu} = (x^\lambda)^\mu$;
2) $y^{-1}x^\lambda y = (y^{-1}xy)^\lambda$,
3)
$$x_1^\lambda x_2^\lambda \cdots x_n^\lambda = t_1^\lambda t_2^{\binom{\lambda}{2}} \cdots t_k^{\binom{\lambda}{c}},$$
where c is the nilpotency class of the group generated by the elements $x_1, x_2, \dots, x_n$, and $t_1, t_2, \dots, t_c$ are Petresco words ([4]), p. 21).

In condition 1), 1 is the identity of the field Ω; the elements x_i, x, y are arbitrary in G, and λ, μ are arbitrary elements of Ω; finally, condition 3) is assumed to hold for every finite n.

An Ω-powered locally nilpotent group G is called an ΩR-powered group if from the equality $x^\lambda = y^\lambda$ for some $\lambda \in \Omega$ it follows that $x = y$. If n is the least number which bounds from above the minimal number of Ω-generators of each finitely generated Ω-powered subgroup of the Ω-powered group G, then we shall say that G has Ω-rank n. The notion of an Ω-homomorphism of one Ω-powered group into another is defined in the natural way.

Theorem 1. In order that a finitely generated solvable torsion-free group Γ have a faithful matrix representation over a field Ω of characteristic zero, it is necessary and sufficient that the group Γ have the structure

$$\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \{e\},$$

where Γ/Γ_1 is a finite group, Γ_1/Γ_2 is a finitely generated abelian group, and the group Γ_2 can be embedded isomorphically in an ΩR-powered nilpotent group H of finite Ω-rank, moreover the restriction of each inner automorphism of the group Γ_1 to the subgroup Γ_2 induces an Ω-automorphism of the group H.

Corollary. In order that a finitely generated solvable torsion-free group Γ have a faithful matrix representation over the field of ra

rational numbers, it is necessary and sufficient that the group Γ have a series

$$\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \{e\},$$

where Γ/Γ_1 is a finite group, Γ_1/Γ_2 is a finitely generated abelian group, and Γ_2 is a torsion-free nilpotent group of finite rational rank.

This corollary gives an answer to the question posed in ([5]).

Theorem 2. In order that a torsion-free solvable group Γ with trivial center be isomorphically represented by matrices over some field Ω of characteristic zero, it is necessary and sufficient that the following conditions be satisfied:

1) in Γ the minimal condition holds for the centralizers of an ascending sequence of subgroups of Γ;
2) in Γ there is a normal series

$$\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \{e\},$$

where Γ/Γ_1 is a finite group, Γ_1/Γ_2 is an abelian group, and the group Γ_2 can be isomorphically embedded in the ΩR-completion of a nilpotent group H of finite Ω-rank, with the restriction of each inner automorphism of the group Γ_1 to the subgroup Γ_2 inducing an Ω-automorphism of the group H.

Using the theorem of A. I. Mal' cev cited above, D. M. Smirnov showed in ([6]) that a finitely generated free solvable group of derived length 3 has no faithful matrix representation over any field. In connection with this there arises the question of the representability of two-step solvable groups by matrices over some field of characteristic zero. It is quite easy to construct an example of a two-step solvable torsion-free group which cannot be faithfully represented by matrices over any field of characteristic zero. In particular, such a group will be the group Γ which is the semidirect product of the direct sum H of a countable number of rational groups H_n $(n = 1, 2, ...)$ and the infinite cyclic group $\{z\}$, where $z^{-1}h_n z = nh_n$ $(h_n \in H_n)$.

Theorem 3. Every finitely generated two-step solvable torsion-free group Γ has a faithful matrix representation over some field of characteristic zero.

Theorem 4. The discrete wreath product $G = \Gamma \operatorname{wr} H$, where Γ and H are torsion-free abelian groups, is isomorphically embeddable in the group of matrices of the second order over some field of characteristic zero.

Theorem 5*. A free nilpotent group of nilpotency class n admits a faithful matrix representation over some field of characteristic zero.

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CITED LITERATURE

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* This theorem was obtained jointly with V. G. Vilyatser.