SOME ESTIMATES FOR A SINGULAR INTEGRAL WITH SUMMABLE DENSITY

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Abstract

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S. K. ABDULLAEV, A. A. BABAEV

SOME ESTIMATES FOR A SINGULAR INTEGRAL WITH SUMMABLE DENSITY

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Let the function $u(x)$ be summable on (a, b) and belong to L_p $(p > 1)$ on every segment $[a+\xi, b-\eta]$ $(\xi, \eta > 0)$. Introduce the functions

$$\Omega_p(u, \xi, \eta) = \left\{ \int_{a+\xi}^{b-\eta} |u(x)|^p \, dx \right\}^{1/p},$$

$$\omega_p(u, \tau, \xi, \eta) := \sup_{h \in A} \left\{ \int_{a+\xi}^{b-\eta-h} |u(x+h) - u(x)|^p \, dx \right\}^{1/p},$$

where $\xi, \eta, \tau > 0$; $\xi + \eta \le b - a = l$; $1 < p \le \infty$; $A = \{h;\ 0 < h \le \min\{\tau, l - \xi - \eta\}\}$.

For $p = +\infty$

$$\Omega_p(u, \xi, \eta) = \max_{x \in [a+\xi, b-\eta]} |u(x)| = \Omega(u, \xi, \eta),$$

$$\omega_p(u, \tau, \xi, \eta) = \max_{\substack{x, y \in [a+\xi, b-\eta] \\ |x-y| \le \tau}} |u(x) - u(y)| = \omega(u, \tau, \xi, \eta).$$

Denote

$$\tilde{u}(x) = \int_a^b \frac{u(s)}{s-x} \, ds = \lim_{\varepsilon \to +0} \left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b \right) \frac{u(s)}{s-x} \, ds.$$

In the present work we consider the question of the relation between the ordered pairs
$(\Omega_p(\tilde{u}, \xi, \eta), \omega_p(\tilde{u}, \tau, \xi, \eta))$ and $(\Omega_p(u, \xi, \eta), \omega_p(u, \tau, \xi, \eta))$.

This problem in the case $p = +\infty$ (in the class of functions continuous on (a, b)) was posed in ([1]) and solved in the works ([1,2]).

Theorem 1. Let $1 < p \leq \infty$. If the integrals

$$\int_0 \frac{\Omega_p(u,t,t)}{t^{1/p}}\,dt, \qquad \int_0 \frac{\omega_p(u,t,\xi/2,\eta/2)}{t}\,dt,$$

converge, then for $0 < \xi, \eta \leq b/2$, $\delta > 0$, the estimates

$$\Omega_p(\tilde{u},\xi,\eta) \leq cq\left\{\int_0^{l/2} \frac{\Omega_p(u,t,l/4)}{t^{1/p}(t+\xi)^{1/q}}\,dt + \int_0^{l/2} \frac{\Omega_p(u,l/4,t)}{t^{1/p}(t+\eta)^{1/q}}\,dt + \right.$$
$$\left. + \int_0^{\xi/2} \frac{\omega_p(u,t,\xi/2,l/4)}{t}\,dt + \int_0^{\eta/2} \frac{\omega_p(u,t,l/4,\eta/2)}{t}\,dt\right\}, \tag{1}$$

$$\omega_p(\tilde{u},\delta,\xi,\eta) \leq cq\left\{\frac{\delta}{\xi+\delta}\int_0^{l/2} \frac{\Omega_p(u,t,l/4)}{t^{1/p}(t+\xi)^{1/q}}\,dt + \frac{\delta}{\eta+\delta}\int_0^{l/2} \frac{\Omega_p(u,l/4,t)}{t^{1/p}(t+\eta)^{1/q}}\,dt + \right.$$
$$\left. +\delta\int_0^{\xi/2} \frac{\omega_p(u,t,\xi/2,l/4)}{t(t+\delta)}\,dt + \delta\int_0^{\eta/2} \frac{\omega_p(u,t,l/4,\eta/2)}{t(t+\delta)}\,dt\right\}, \tag{2}$$

where $1/p + 1/q = 1$; c is a constant depending only on l.

Note that for $p = +\infty$ these estimates turn into the estimates of the work ([2]), which is a refinement and development of the work ([1]).

Let us consider some constructions based on the preceding estimates. Denote by G ([2]) the set of ordered pairs of functions $(\varphi(\xi,\eta), \psi(\delta,\xi,\eta))$, defined respectively on $\{0 < \xi, \eta \mid \xi+\eta \leq l\}$, $\{0 < \delta, \xi, \eta \mid \delta+\xi+\eta \leq l\}$, and satisfying the conditions:

1) $\varphi(\xi,\eta)$, $\psi(\delta,\xi,\eta)/\delta$ are positive and almost decreasing* in each of the arguments uniformly with respect to the others;
2) $\lim\limits_{\delta\to+0} \psi(\delta,\xi,\eta) = 0$.

By definition, a function $u(x)$, given on (a, b), belongs to the set $H^p_{\varphi\psi}$ if there exist constants $c_1(u), c_2(u) > 0$ such that

$$\Omega_p(u,\xi,\eta) \leq c_1(u)\varphi(\xi,\eta), \qquad \omega_p(u,\delta,\xi,\eta) \leq c_2(u)\psi(\delta,\xi,\eta),$$

where $(\varphi,\psi) \in G$.

By introducing the norm

$$\|u\|_{\varphi\psi}^p = \max\left\{\sup_{\xi,\eta}\frac{\Omega_p(u,\xi,\eta)}{\varphi(\xi,\eta)},\ \sup_{\xi,\eta,\delta}\frac{\omega_p(u,\delta,\xi,\eta)}{\psi(\delta,\xi,\eta)}\right\},$$

$H_{\varphi\psi}^p$ is turned into an infinite-dimensional Banach space.

Theorem 2. Let $(\varphi_1,\psi_1),(\varphi_2,\psi_2)\in G$.

Then:

a) if $\varphi_1\sim\varphi_2$, $\psi_1\sim\psi_2$, then $H_{\varphi_1\psi_1}^p$ and $H_{\varphi_2\psi_2}^p$ coincide**;

b) if the limiting relations

$$\lim_{\delta\to+0}\frac{\psi_1(\delta,\xi,\eta)}{\psi_2(\delta,\xi,\eta)}=0,\qquad \lim_{\xi\to+0}\frac{\psi_1(\delta,\xi,\eta)}{\psi_2(\delta,\xi,\eta)}=0,$$

$$\lim_{\eta\to+0}\frac{\psi_1(\delta,\xi,\eta)}{\psi_2(\delta,\xi,\eta)}=0,\qquad \lim_{\xi\to+0}\frac{\varphi_1(\xi,\eta)}{\varphi_2(\xi,\eta)}=0,\qquad \lim_{\eta\to+0}\frac{\varphi_1(\xi,\eta)}{\varphi_2(\xi,\eta)}=0,$$

are satisfied uniformly, then $H_{\varphi_1\psi_1}^p$ is a proper part of $H_{\varphi_2\psi_2}^p$, and the embedding is completely continuous.

Denote by Φ the set of ordered pairs of functions $(\varphi,\psi)\in G$ satisfying the conditions:

1) $\psi(\delta,\xi,\eta)$ almost increases with respect to δ;

2) $\psi(\delta_1+\delta_2,\xi,\eta)=O(\psi(\delta_1,\xi,\eta)+\psi(\delta_2,\xi,\eta))$***;

3) $\psi(\delta,\xi,\eta)=O(\varphi(\xi,\eta))$.

Following ([2]), introduce the set H_p of ordered pairs of functions $(\varphi(\xi),\psi(\delta,\xi))$ satisfying the conditions:

1) $\varphi(\xi)>0,\ \psi(\delta,\xi)>0$;

2)

$$\int_0^{1/2}\frac{\varphi(t)}{t^{1/p}(t+\xi)^{1/q}}\,dt=O(\varphi(\xi));$$

3)

$$\delta\int_0^{\xi}\frac{\psi(t,\xi/2)}{t(t+\delta)}\,dt=O(\psi(\delta,\xi));$$

4)

$$\frac{\delta}{\xi+\delta}\varphi(\xi) = O(\psi(\delta,\xi)).$$

By definition $(\varphi,\psi) \in \Phi H_p$ if $(\varphi,\psi) \in \Phi$ and $(\varphi(\xi, l/4), \psi(\delta,\xi,l/4))$, $(\varphi(l/4,\eta), \psi(\delta, l/4, \eta)) \in H_p$.

Theorem 3. Let $(\varphi,\psi) \in \Phi H_p$. Then the operator

$$Au = \int_a^b \frac{u(s)}{s-x}\,ds$$

acts in $H^p_{\varphi\psi}$ and is bounded.

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* A nonnegative function $f(x)$, defined on a set $\chi \subset (-\infty, +\infty)$, is called almost increasing (almost decreasing) if there exists a constant $c > 0$ such that the inequality $x_1 \le x_2$, $x_1, x_2 \in \chi$, implies the inequality $f(x_1) \le cf(x_2)$ $(f(x_1) \ge cf(x_2))$.

** Nonnegative functions $f(x)$ and $g(x)$, defined on χ, are called equivalent $(f \sim g)$ if there exist constants $B_1, B_2 > 0$ such that for every $x \in \chi$ the inequalities $B_1 f(x) \le g(x) \le B_2 f(x)$ hold.

*** Here and in what follows, uniform satisfaction of the O-relation is assumed.

This theorem, in the case $p = +\infty$, was proved in [2].
It is easy to verify that the pair of functions

$$\varphi(\xi,\eta) = \frac{1}{\xi^\alpha} + \frac{1}{\eta^\beta}, \qquad \psi(\delta,\xi,\eta) = \frac{\delta^\alpha}{\xi^\alpha(\xi+\delta)^\alpha} + \delta^\gamma + \frac{\delta^\beta}{\eta^\beta(\eta+\delta)^\beta}$$

$$(0 < \alpha, \beta < 1/q;\ 0 < \gamma < 1)$$

belongs to ΦH_p.

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Azerbaijan State University
named after S. M. Kirov
Baku

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Note: Figure translations are in progress. See original paper for figures.