SEMI-ABELIAN CATEGORIES

MATHEMATICS

1969

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Abstract

Full Text

UDC 519.48

MATHEMATICS

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SEMI-ABELIAN CATEGORIES

(Presented by Academician P. S. Novikov on 3 III 1969)

In this note a class of categories is described which includes all abelian categories and in which the basic properties of the latter used in homological algebra are preserved, while at the same time it contains such non-abelian categories occurring in topological algebra, functional analysis, and algebra as the categories of (all or only separable) topological commutative groups, topological linear spaces and locally convex spaces, the category of Banach spaces, the category of filtered modules over filtered rings ([1]), etc.

We shall call an *embedding* any monomorphism α such that every morphism β for which $\varphi\alpha = \psi\alpha$ always implies $\varphi\beta = \psi\beta$, majorizes α, i.e. has the form $\beta = \alpha\gamma$; epimorphisms with the dual property will be called *quotient maps.** Subobjects and quotient objects (in the sense of Grothendieck ([5])) corresponding to embeddings and quotient maps will be called *strict.* By the *image* and *strict image* of a morphism α we shall mean (defined up to monoequivalence) the least monomorphism $\operatorname{im}\alpha$ majorizing it and the embedding $\operatorname{im}_s\alpha$, as well as the corresponding subobject $\operatorname{Im}\alpha$ and strict subobject $\operatorname{Im}_s\alpha$; dually we define *coimages* and *strict coimages.* A morphism which is both a mono- and an epimorphism will be called a *bimorphism.*

I. A category $\mathcal{C}$ satisfying axioms SA1-SA6 is called a *semi-abelian category.* After each axiom, beginning with SA3, its principal consequences are given (relying also on the preceding axioms); the dual propositions (also true), as a rule, are not formulated.

SA1. $\mathcal{C}$ contains a zero object.

SA2. Every morphism in $\mathcal{C}$ has a kernel and a cokernel.

SA3. For every morphism $\alpha : A \to B$ in $\mathcal{C}$, the morphism $\dot{\alpha}$ in the canonical decomposition

$$A \xrightarrow{\alpha} B = A \to \operatorname{Cok}\ker\alpha \xrightarrow{\dot{\alpha}} \operatorname{Ker}\operatorname{cok}\alpha \to B \tag{1}$$

(which follows from SA2) is a bimorphism.

1. Every morphism in $\mathcal{C}$ has an image, strict image, coimage, and strict coimage; namely,

$$\ker \operatorname{cok} \alpha \cdot \alpha = \operatorname{im} \alpha, \qquad \ker \operatorname{cok} \alpha = \operatorname{im}_s \alpha,$$

and

$$\alpha \operatorname{cok} \ker \alpha = \operatorname{coim} \alpha, \qquad \operatorname{cok} \ker \alpha = \operatorname{coim}_s \alpha.$$

2. α is an embedding (then and) only then, when $\alpha = \ker \operatorname{cok} \alpha$. α is a monomorphism then (and only then) when $\operatorname{Ker} \alpha = 0$.
3. The composition of embeddings is an embedding. If $\alpha = \beta\gamma$ and α is an embedding, then γ is an embedding.
4. For any embedding $\alpha : A \to C$ and morphism $\beta : B \to C$ there exists a couniversal square

$$\begin{tikzcd} D \arrow[r, "\varphi"] \arrow[d, "\psi"'] & B \arrow[d, "\beta"] \\ A \arrow[r, "\alpha"] & C \end{tikzcd} \tag{2}$$

where $\varphi(= \ker \operatorname{cok} \alpha \cdot \beta)$ is an embedding. In any co-universal square (2), φ is a monomorphism if (and only if) α is a monomorphism.

5. The ordered classes of strict subobjects and strict factor objects of each object are inversely isomorphic lattices.

A morphism α in $\mathcal{C}$ will be called **strict**, or also a **homomorphism**, if the morphism $\dot{\alpha}$ associated with it in (1) is an isomorphism, or, equivalently, if $\alpha = \beta\gamma$, where β is an embedding and γ is an epimorphism. The sequence

$$\cdots \xrightarrow{\alpha} A \xrightarrow{\beta} \cdots$$

will be called exact (coexact) at A if $\operatorname{Im} \alpha = \operatorname{Ker} \beta$ ($\operatorname{Coim} \beta = \operatorname{Cok} \alpha$), and simply (co)exact if it is (co)exact at every interior vertex. Thus, in an exact sequence all morphisms, except possibly the last, are strict. In particular,

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is an exact sequence if and only if $\alpha = \ker \beta$ and $\beta = \operatorname{cok} \alpha$.

SA4. Let

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d, "\alpha"'] & B_1 \arrow[r] \arrow[d, "\beta"'] & C_1 \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd} \tag{3}$$

be a commutative diagram in $\mathcal{C}$ with exact rows. If α and γ are isomorphisms, then β is an isomorphism.

* These notions were first introduced in (2) and then independently in (3,4).

6. Let

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\alpha"'] & B_1 \arrow[r] \arrow[d, "\beta"'] & C_1 \arrow[r] \arrow[d, "\gamma"'] & D_1 \arrow[r] \arrow[d, "\delta"'] & E_1 \arrow[d, "\varepsilon"'] \\
A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & D_2 \arrow[r] & E_2
\end{tikzcd}$$

   be a commutative diagram with exact rows. If α is an epimorphism, ε is a monomorphism, and β and δ are isomorphisms, then γ is an isomorphism.

SA5. Each ordered pair of objects in $\mathcal{C}$ has a product (and hence also a sum (^{6})).

7. Every pair of morphisms $\varphi, \psi : X \to Y$ has an equalizer and a coequalizer. For any morphisms $\alpha : A \to C$ and $\beta : B \to C$ there exists a co-universal square (2).

8. Any subobjects (A, α) and (B, β) of an object C have an intersection $A \cap B$ (represented by the diagonal DC of the co-universal square (2), and strict if α and β are embeddings) and a union $A \cup B$ (which is the image of the morphism $(\alpha, \beta) : A + B \to C$).

9. ("Second isomorphism theorem.") If (A, α) and (B, β) are strict subobjects of an object C and $\operatorname{cok} \beta \cdot \alpha$ is a homomorphism, then $A \cup B / B$ is canonically isomorphic to $A / A \cap B$, and $(\alpha, \beta) : A + B \to C$ is a homomorphism, so that $\operatorname{im}(\alpha, \beta) : A \cup B \to C$ is an embedding.

10. $\mathcal{C}$ is an additive category (in the sense of Grothendieck (^{5})).

SA6. If α and β are homomorphisms and $\operatorname{Im} \alpha \supset \operatorname{Ker} \beta$, then $\beta\alpha$ is a homomorphism.

Since axioms SA1—SA6 are self-dual, the category dual to a semi-abelian one is semi-abelian.

11. ("First isomorphism theorem.") If A is a strict subobject of an object B, and B is a strict subobject of an object C, then the canonical morphism

$$B/A \to C/A$$

   is an embedding, and $C/A \,/\, B/A$ is canonically isomorphic to C/B.

12. Let α and β be embeddings with common codomain. If $\operatorname{cok} \alpha \cdot \beta$ is an embedding (an epimorphism, a homomorphism), then the same is true for $\operatorname{cok} \beta \cdot \alpha$.

13. The nine lemma is valid.

Thus the strict morphisms of a semi-abelian category satisfy the requirements imposed in (^{7}) on "proper" morphisms, so that every semi-abelian category is an "abelian category" in the sense of Heller, in which all short exact sequences are proper.

14. If in the co-universal square (2) α is an epimorphism, then φ is also an epimorphism.

15. If in the commutative diagram

$$\begin{tikzcd}
 & A' \arrow[r] \arrow[d, "\alpha'"'] & A \arrow[r] \arrow[d, "\alpha"'] & A'' \arrow[r] \arrow[d, "\alpha''"'] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' &
\end{tikzcd}$$

the rows are exact, and α', α, and α'' are homomorphisms, then there exists a morphism

$$\operatorname{Ker}\alpha'' \to \operatorname{Cok}\alpha'$$

such that the sequence

$$\operatorname{Ker}\alpha' \to \operatorname{Ker}\alpha \to \operatorname{Ker}\alpha'' \to \operatorname{Cok}\alpha' \to \operatorname{Cok}\alpha \to \operatorname{Cok}\alpha''$$

(where the remaining morphisms are canonical) is exact.

II. The definitions of "relations," their composition, ordering, and involution given in (⁶) carry over verbatim to semiabelian categories. In this case the relations over a semiabelian category $\mathcal{C}$ form an I-category $\mathfrak{R}(\mathcal{C})$, satisfying all of Puppe' s axioms K1-K6 except K3b, and the functor $\varphi : \mathcal{C} \to \mathfrak{R}(\mathcal{C})$, assigning to each morphism $\alpha : A \to B$ in $\mathcal{C}$ the image of the morphism

$$\begin{pmatrix} 1_A \\ \alpha \end{pmatrix} : A \to A \times B,$$

maps each set $\operatorname{Hom}(A, B)$ bijectively onto the set of all proper relations from A to B, preserving kernels and images. The proofs of these assertions can be carried out by replacing, in the arguments given for abelian categories in (⁶), epimorphisms by quotient maps and using the following proposition (which follows from 1 and 14):

16. If α and β are morphisms with common codomain, then

$$\operatorname{Im}\alpha \subset \operatorname{Im}\beta$$

if and only if there exist a quotient map ν and a morphism ω such that

$$\alpha\nu = \beta\omega.$$

Moreover, the proofs are considerably simplified if one also uses the following property of the composition of relations (cf. (⁸)): let u and v be relations from A' to C and from C to B', given as images of the morphisms

$$\begin{pmatrix} \alpha' \\ \alpha \end{pmatrix} : A \to A' \times C \quad \text{and} \quad \begin{pmatrix} \beta \\ \beta' \end{pmatrix} : B \to C \times B';$$

then

$$v \circ u = \operatorname{Im} \begin{pmatrix} \alpha'\psi \\ \beta'\varphi \end{pmatrix},$$

where φ and ψ are morphisms from the couniversal square (2).

III. Semiabelian categories are close to the "special preabelian categories" introduced in (^{9}). The authors of that work call an additive category satisfying axioms SA2-SA3 preabelian, and a special preabelian category is a preabelian category in which, together with its dual duplicate, the following "abelian" analogue of Proposition 14 holds:

14′. If in the couniversal square (2) α is an epimorphism, then φ is also an epimorphism.

By 10, every semiabelian category is "preabelian." However, 14′ is not true in all semiabelian categories; thus, because of the existence of non-surjective epimorphisms, this proposition is false in the category of separated topological vector spaces. But the entire difference between Propositions 14 and 14′ is precisely the difference between semiabelian categories and "special preabelian" ones:

17. A category $\mathscr{C}$ is semiabelian if and only if $\mathscr{C}$ is an additive category in which axioms SA2-SA3 and Proposition 14 hold.

In (^{9}) a functor S is constructed which assigns to each object X of a special preabelian category the class X_s of all its strict subobjects, and to each morphism $\alpha : X \to Y$ the map $\alpha_s : X_s \to Y_s$ by the rule

$$\alpha_s(\operatorname{Im}_s \xi) = \operatorname{Im}_s(\alpha\xi).$$

Together with 14′, this makes it possible to prove by "diagram chasing" a number of propositions in which, however, exactness is understood in a weakened sense: a sequence

$$\cdots \xrightarrow{\alpha} A \xrightarrow{\beta} \cdots$$

is called exact at A if

$$\operatorname{Im}_s \alpha = \operatorname{Ker} \beta;$$

because of this, for example, the "isomorphism theorems" turn out in fact to be only theorems on bimorphisms.* Validity in semiabelian categories Proposition 14 makes it possible to get rid of this drawback by replacing the functor S by the functor I, which assigns to each object X the class X_i of all its subobjects, and to each morphism $\alpha : X \to Y$ the mapping $\alpha_i : X_i \to Y_i$ according to the rule $\alpha_i(\operatorname{Im} \xi) = \operatorname{Im}(\alpha\xi)$ (^{10}). Thus, starting from the definition of a semiabelian category contained in Proposition 17, one can also obtain the preceding diagrammatic propositions by the method of "diagram chasing," analogous to that developed in (^{10,9}).

I express my gratitude to A. Merzon for the assistance rendered in checking the semiabelianness of the nonabelian categories noted at the beginning of the note.

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* We note that the proof of the analogue of Proposition 15 in (^{9}) is incorrect, and it is unclear whether the assertion itself is true.

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Received
1 III 1969

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Note: Figure translations are in progress. See original paper for figures.