AN ITERATIVE METHOD FOR SOLVING AN INTEGRAL EQUATION OF POTENTIAL THEORY

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Abstract

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AN ITERATIVE METHOD FOR SOLVING AN INTEGRAL EQUATION OF POTENTIAL THEORY

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The main results of the present paper are formulated in Theorems 1, 2, 3. Theorem 1 is of a general character. Theorems 2 and 3 represent an application of Theorem 1 to two principal problems of electrostatics, namely to the determination of the density of surface charges in two cases: (a) a conductor in a given external field and (b) a conductor with a given total charge.

§ 1. Let A be a completely continuous linear operator. It is known that the series

$$(I-\lambda A)^{-1}=\sum_{n=0}^{\infty}\lambda^n A^n \tag{1}$$

converges in the uniform operator topology if $|\lambda|<r==1/\lim\limits_{n\to\infty}\sqrt[n]{\|A^n\|}$, where r is the distance from the origin to the nearest eigenvalue of the operator A [1]. Here eigenvalues are understood in the sense adopted in the theory of integral equations: $\varphi_n=\lambda_n A\varphi_n$. Suppose that for $\lambda=\lambda_1$, $|\lambda_1|=r$, the operator A has an eigenvalue to which there corresponds a finite-dimensional eigensubspace H_1. We shall assume that the operator A acts in a Hilbert space H, although the arguments, with minor changes, remain valid also in the case when A acts in a Banach space.

Denote by G_1 the eigensubspace of the adjoint equation

$$(I-\overline{\lambda}_1 A^*)\psi_j=0,\qquad G_1=\{\psi_j\}. \tag{2}$$

It is known that, for the solvability of the equation

$$(I-\lambda_1 A)\varphi=f \tag{3}$$

it is necessary and sufficient that $f \subset G_1^{\perp}$. By $G_1^{\perp}$ is denoted the orthogonal complement in H to the subspace G_1. The subspace $G_1^{\perp}$ is invariant with respect to the operator A. Suppose that the solution of equation (3) belonging to $G_1^{\perp}$ is unique. This assumption means that the eigensubspace corresponding to the number λ_1 coincides with the root subspace corresponding to the number λ_1 [2]. An equivalent assumption is that λ_1 is a simple pole of the resolvent $(I-\lambda A)^{-1}$. We shall show that then the operator A has in $G_1^{\perp}$ a first eigenvalue outside the circle $|\lambda| = r$, and therefore the series (1) converges in the sense of uniform convergence of operators in $G_1^{\perp}$. Since $G_1^{\perp} \subset H$, it is enough to show that $\lambda = \lambda_1$ is not an eigenvalue of the operator A, considered as an operator in $G_1^{\perp}$. Assuming the contrary, suppose that $(I - \lambda_1 A)\varphi = 0$, $\varphi \subset G_1^{\perp}$. By the assumption made, $\varphi = 0$, as was required to be shown. We formulate the preceding observation as a theorem.

Theorem 1. *Let A be a completely continuous operator acting in a Hilbert space H; let λ_1 be its eigenvalue of smallest modulus; let G_1 be the subspace of zeros of the operator $I - \overline{\lambda}_1 A^*$; let $G_1^{\perp}$ be its*

orthogonal complement in H. If the proper subspace of the operator A_1 corresponding to the eigenvalue λ_1 coincides with the root subspace of the operator A corresponding to this same number, then the operator A_1, considered in the invariant subspace $G_1^{\perp}$, has eigenvalues outside the circle of radius $r = 1/\lim\limits_{n\to\infty} \sqrt[n]{\|A^n\|}$, and therefore equation (3), considered as an equation in $G_1^{\perp}$*, is solvable, and in fact uniquely so, by the iteration method:

$$\varphi_{n+1} = \lambda_1 A\varphi_n + f; \qquad \varphi_0 = \Phi \tag{4}$$

for any $f \subset G_1^{\perp}$ and any $\Phi \subset G_1^{\perp}$, and the solution does not depend on the initial approximation $\Phi \subset G_1^{\perp}$.

Remark. If the value $|\lambda_2|$ is known, where λ_2 is the second eigenvalue of the operator A (it is assumed that the eigenvalues are arranged in increasing order of modulus), then the process (4) converges no more slowly than a geometric progression with $q = |\lambda_1/\lambda_2|$. (If Γ is a sphere, then $\lambda_2 = 3$, $q = 1/3$ for the operator A defined by formula (5), see below.)

§ 2. Let us consider the application of Theorem 1 to a problem of potential theory that is of interest in electrostatics. If an ideally conducting body with piecewise-smooth surface Γ is placed in an arbitrary external electrostatic field $E_0 = -\nabla\varphi$, then the density σ of the charges induced on the surface of the conductor satisfies the integral equation

$$\sigma(s) = -2\frac{\partial\varphi}{\partial n_s} - \int_{\Gamma} \frac{\partial}{\partial n_s}\frac{1}{2\pi r_{st}}\sigma(t)\,dt \equiv f - A\sigma, \tag{5}$$

where n is the exterior normal to the surface Γ; dt is the area element of this surface. The operator A in equation (5) is completely continuous in $H = L_2(\Gamma)$

and, as is known ([4]), has no eigenvalues inside the unit circle. The number $\lambda_1 = -1$ is an eigenvalue of the operator A. Equation (2) in our case has the form

$$\psi(s) + \int_\Gamma \frac{\partial}{\partial n_t} \frac{1}{2\pi r_{st}} \psi(t)\, dt = 0 \tag{6}$$

and, as is well known, has the single eigenfunction $\psi = 1$ ([4], p. 624). Consequently, the subspace $G_1^\perp$ in the problem under consideration consists of functions σ satisfying the condition

$$\int_\Gamma \sigma(t)\, dt = 0. \tag{7}$$

Let us note at once that

$$\int_\Gamma \frac{\partial \varphi}{\partial n_s}\, ds = 0, \tag{8}$$

since the potential of the external electrostatic field is a harmonic function. The homogeneous equation (5), as is known ([4], p. 625), has one and only one linearly independent solution φ_0 (the solution of Robin' s problem). It is known, moreover, that this solution satisfies the condition ([4], p. 625)

$$\int_\Gamma \varphi_0(s)\, ds \neq 0. \tag{9}$$

This means that every solution of the homogeneous equation (5) belonging to $G_1^\perp$ (i.e., satisfying condition (7)) is identically zero. All the conditions of Theorem 1 are fulfilled. Therefore the following is true.

Theorem 2. *The solution of the problem of finding the distribution of charges induced on an ideally conducting body placed in an electrosta-*

* For this equation to be regarded as an equation in $G_1^\perp$, it is necessary and sufficient that $f \subset G_1^\perp$, since the

static external field $E_0 = -\nabla\varphi$, where φ is the potential of the external field, can be obtained by means of the convergent iterative process according to the scheme

$$\sigma_{n+1}(s) = -\int_\Gamma \frac{\partial}{\partial n_s} \frac{1}{2\pi r_{st}}\, \sigma_n(t)\, dt - 2\frac{\partial \varphi(s)}{\partial n_s}. \tag{10}$$

As the initial distribution of charges one may take any distribution satisfying condition (7), for example $\sigma_0(s) = -2\partial\varphi/\partial n_s$.

Theorem 2 gives a convergent iterative method for solving certain problems of electrostatics (for example, the problem of a cylinder in an external field), which were previously solved by a numerical method based on solving systems of algebraic equations arising in the investigation of a certain integral equation of the first kind (5).

§ 3. The problem of the distribution of charge q on the surface of a perfectly conducting body leads to the equations

$$\sigma(s) = -A\sigma; \qquad \int_\Gamma \sigma(t)\,dt = q, \tag{11}$$

where σ is the surface charge density, and the operator A is defined in equality (5). Solving problem (11) is equivalent to solving the Robin problem ([4]). We shall give a solution of problem (11) by means of a convergent iterative procedure. Introduce the notation: $H = L_2(\Gamma)$; G_1 is the one-dimensional subspace whose elements are functions that are constant on the surface Γ. In accordance with the formula $H = G_1 \oplus G_1^\perp$, represent the function σ in the form

$$\sigma = \sigma_1 + \sigma_2; \qquad \sigma_2 \subset G_1^\perp; \qquad \sigma_1 \subset G_1. \tag{12}$$

Since $\int_\Gamma \sigma_2 dt = 0$, $\sigma_1 = \text{const}$, from the second condition (11) we obtain

$$\sigma_1 = q/S, \qquad S \equiv \operatorname{mes}\Gamma. \tag{13}$$

Consequently,

$$\sigma(t) = q/S + \sigma_2(t); \qquad \int_\Gamma \sigma_2(t)\,dt = 0. \tag{14}$$

Equation (11), after substituting expression (14) for $\sigma(t)$, will take the form

$$\sigma_2 = -A\sigma_2 + f; \qquad f \equiv -\frac{q}{S} - \frac{q}{S}\int_\Gamma \frac{\partial}{\partial n_s}\frac{1}{2\pi r_{st}}\,dt. \tag{15}$$

We note that

$$\int_\Gamma f\,ds = -\frac{q}{S}\left[S + \int_\Gamma ds \int_\Gamma \frac{\partial}{\partial n_s}\frac{1}{2\pi r_{st}}\,dt\right]$$

$$= -\frac{q}{S}\left[S + \int_\Gamma dt \int_\Gamma ds\,\frac{\partial}{\partial n_s}\frac{1}{2\pi r_{st}}\right] = -\frac{q}{S}[S - S] = 0. \tag{16}$$

As was already noted above, the homogeneous equation (15) has in $G_1^\perp$ only the zero solution. According to formula (16), $f \subset G_1^\perp$. Considering equation (15) in $G_1^\perp$ and applying Theorem 1, all conditions of which are fulfilled, we obtain the theorem:

Theorem 3. The distribution of surface charge density on a perfectly conducting body with piecewise-smooth surface Γ can be found by formula (14), where the function σ_2 is uniquely determined by means of the convergent iterative process

$$\sigma_2^{(n+1)} = -A\sigma_2^{(n)} + f; \qquad \sigma_2^{(0)} = f, \tag{17}$$

where the function f is defined by formula (15).

Remark. Theorem 3 contains, in particular, a theoretical justification of the well-known empirical Houw method ([3]) for calculating the capacitance of conductors.

In fact, according to Hoy, the capacitance is computed approximately by the formula

$$C = \frac{q}{V}, \qquad \text{where} \quad V = \frac{1}{S}\int_\Gamma ds \int_\Gamma \frac{q}{S}\frac{dt}{4\pi r_{st}} = \frac{q}{S^2}\int_\Gamma ds \int_\Gamma \frac{dt}{4\pi r_{st}}.$$

The indicated expression for V is obtained if, in formula (14), the term σ_2 is neglected and the potential produced by the part σ_1 of the surface-charge distribution is averaged over the surface. It follows from this that Hoy's method is applicable in those cases in which the potential produced on the surface of the body by charges distributed with surface density σ_2 is noticeably smaller than the potential produced by charges distributed with surface density $\sigma_1 = q/S$. In conclusion, let us note that, although the literature devoted to the method of iterations for the solution of functional equations is extensive (see, for example, paper [6] and the references cited therein), the simple idea constituting the content of Theorem 1 has, as far as the author knows, not been encountered before.

§ 4. From the results of § 2, the definition of the dipole moment

$$P_i = \int_\Gamma s_i \sigma(s)\, ds$$

and of the polarizability tensor α_{ij}, $P_i = \alpha_{ij} V E_{0j}$, there follows an approximate formula for computing this tensor for a body of arbitrary shape placed in a homogeneous electrostatic field E_0:

$$\alpha_{ij} = 4\delta_{ij} - \frac{1}{\pi V}\iint_{\Gamma\Gamma} \frac{n_i(s)n_j(t)}{r_{st}}\, ds\, dt; \qquad n_i(s) = \cos(n(s), x_i). \tag{18}$$

An analogous formula for the tensor β_{ij} of magnetic polarizability ($M_i = \beta_{ij} V H_{0j}$) has the form

$$\beta_{ij} = -\frac{1}{\pi V} \iint_{\Gamma\Gamma} \frac{n_i(s) n_j(t)}{r_{st}} \, ds \, dt. \tag{19}$$

Both formulas are valid with error $|1/\lambda_2|^2$, since $\lambda_1 = 1$ in the cases under consideration.

For a dielectric body with constant ε_i, situated in a medium with constant ε_e, the polarizability tensor in a homogeneous electrostatic field E_0 is computed by the approximate formula

$$\alpha_{ij} = \delta_{ij}(2\varepsilon + 2\varepsilon^2) - \frac{\varepsilon^2}{\pi V} \iint_{\Gamma\Gamma} \frac{n_i(s) n_j(t)}{r_{st}} \, ds \, dt; \qquad \varepsilon \equiv \frac{\varepsilon_i - \varepsilon_e}{\varepsilon_i + \varepsilon_e}. \tag{20}$$

The error of this formula is $|\varepsilon/\lambda_2|^2$. Formulas (18), (19) are obtained from formula (20), respectively, for $\varepsilon_i = \infty$, $\varepsilon_i = 0$ (a conductor in electrostatic and a superconductor in magnetostatic fields).

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Note: Figure translations are in progress. See original paper for figures.