EFFECTIVE TRAJECTORIES OF A DYNAMIC MODEL OF PRODUCTION

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EFFECTIVE TRAJECTORIES OF A DYNAMIC MODEL OF PRODUCTION

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The note studies the properties of certain classes of trajectories of a positively homogeneous dynamic model of production. This model may be regarded, on the one hand, as a generalization of the discrete model with variable technology ([1,2]) (and, consequently, of the models of Neumann ([3]) and Gale ([4])); on the other hand, as a generalization of the problem of continuous convex programming ([5]) (and, consequently, of Bellman' s "bottleneck" problem ([6]), and of the problem of continuous linear programming ([7])).

1°. Below we use the following definitions.

1) Let X_i be normed spaces, partially ordered respectively by closed, convex, reproducing, salient cones K_i $(i = 1, 2)$. A mapping* a of the cone K_1 into the collection of subsets of the cone K_2 is called superlinear if

$$a(0) = 0; \qquad \bigcup_{x \in K_1} a(x) = K_2; \qquad a(x_1 + x_2) \supset a(x_1) + a(x_2) \quad (x_1, x_2 \in K_1);$$

$$a(\lambda x) = \lambda a(x) = \lambda a(x) \quad (\lambda > 0,\ x \in K_1);$$

and from the relations $x_n \in K_1,\ x_n \to x,\ y_n \in a(x_n),\ y_n \to y$ it follows that $y \in a(x)$.

2) The normal hull of a mapping a will mean the mapping na which assigns to each x in K_1 the set $na(x)$, which is the union of the conic intervals $\langle 0, y \rangle$ over all $y \in a(x)$.

3) If a is a superlinear mapping, then on the cone K_2 one may define the mapping $\hat{a}$, inverse to the mapping a: for $y \in K_2$,

$$\hat{a}(y) = \{x \in K_1 \mid y \in a(x)\}.$$

4) If a_i $(i = 1, 2)$ is a mapping of the set Y_i into the collection of subsets of the set Y_{i+1}, then the product of the mappings a_1 and a_2 is understood to be the mapping $a_2 \circ a_1$, defined on Y_1 by the formula

$$a_2 \circ a_1(x) = \bigcup_{y \in a_1(x)} a_2(y) \qquad (x \in Y_1).$$

5) Let ξ be a subset of the vector space X; a point $x \in X$ will be called a boundary-below point of the set ξ if $x \in \xi$ and $\lambda x \notin \xi$ for any $\lambda < 1$.

2°. In the model defined here, the time parameter t ranges over a set E of nonnegative real numbers, more than one point in size, containing zero. To each $t \in E$ we assign a finite-dimensional real vector space X_t, partially ordered by a convex closed, reproducing, salient cone K_t. In addition, to each pair of indices $t, \tau \in E,\ \tau > t$, we assign a mapping $a_{\tau,t}$ of the cone K_t into the collection of subsets of K_τ. If the mappings $a_{\tau,t}$ are superlinear and, moreover,

$$a_{t,t'} \circ a_{t',t''} = a_{t,t''} \qquad (t > t' > t'';\ t, t', t'' \in E),$$

then

* Here and in what follows we assume that the mappings under consideration carry each point of their domain into a nonempty subset.

We shall call an object $\mathfrak{M} = \{E,\ (X_t)_{t\in E},\ (K_t)_{t\in E},\ (a_{\tau,t})_{t,\tau\in E;\ \tau>t}\}$ a positively homogeneous dynamic model of production (abbreviated p.h.m.).

Lemma 1. If $\mathfrak{M}$ is a p.h.m., $\tau, t \in E,\ \tau > t,\ x \in K_t$, then the set $a_{\tau,t}(x)$ is convex, closed, and bounded.

A trajectory of a p.h.m. $\mathfrak{M}$ will mean a family $\chi = (x_t)_{t\in E}$ such that $x_t \in K_t$ $(t \in E)$; $x_\tau \in a_{\tau,t}(x_t)$ $(\tau, t \in E;\ \tau > t)$.

Theorem 1. If $\mathfrak{M}$ is a p.h.m., $t', t'' \in E$, $t' > t''$, $x' \in K_{t'}$, $x'' \in a_{t'',t'}(x')$, then there exists a trajectory $(x_t)_{t\in E}$ of the model $\mathfrak{M}$ such that $x_{t'} = x'$, $x_{t''} = x''$.

We shall say that a trajectory $(x_t)_{t\in E}$ issues from the point x if $x_0 = x$. Theorem 1 guarantees that trajectories issue from any point of the cone K_0.

3°. Consider a p.h.m. $\mathfrak{M}$. Put $T = \sup E$. We shall call the model $\mathfrak{M}$ finite if $T \in E$. If $x \in K_0$, then by the symbol Γ_T^x we denote the face of the cone K_T generated by the set $a_{T,0}(x)$ (Γ_T^x coincides with the conical hull of the set $na_{T,0}(x)$; if x is an interior point of K_0, then $\Gamma_T^x = K_T$). A trajectory $\bar{\chi} = (\bar{x}_t)_{t\in E}$ of a finite p.h.m. will be called efficient if there exists a functional $f \neq 0$ from the space $(\Gamma_T^x - \Gamma_T^x)^*$, positive on Γ_T^x, and such that

$$f(\bar{x}_T) = \max_{y \in a_{T,0}(x)} f(y) \quad (\text{here } x = \bar{x}_0).$$

Lemma 2. Let $\mathfrak{M}$ be a finite p.h.m. A trajectory $\bar{\chi} = (\bar{x}_t)_{t\in E}$ of the model $\mathfrak{M}$ is efficient if and only if $\bar{x}_0$ is a lower boundary point of the set $na_{T,0}(\bar{x}_T)$ (here $T = \sup E$).

Let us now consider an arbitrary (not necessarily finite) p.h.m. $\mathfrak{M}$.

For each trajectory $\chi = (x_t)_{t\in E}$ of the model $\mathfrak{M}$ put

$$\hat{a}(\chi) = \bigcup_{t\in E} \widehat{na}_{t,0}(x_t).$$

(If $\mathfrak{M}$ is finite, then $\hat{a}(\chi) = \widehat{na}_{T,0}(x_T)$.) A trajectory $\bar{\chi} = (\bar{x}_t)_{t\in E}$ of the model $\mathfrak{M}$ will be called weakly efficient if * there exists $f \neq 0$ from K_0^* such that

$$f(\bar{x}_0) = \min_{x\in\hat{a}(\bar{\chi})} f(x);$$

efficient if $\bar{x}_0$ is a lower boundary point of the set $\hat{a}(\bar{\chi})$; strongly efficient if there exists $f \in K_0^*$ such that

$$f(\bar{x}_0) = \min_{x\in\hat{a}(\bar{\chi})} f(x) > 0.$$

It is clear that a strongly efficient trajectory is efficient; in turn, an efficient trajectory is weakly efficient. At the same time, a weakly efficient trajectory issuing from an interior point of the cone K_0 will be strongly efficient.

Theorem 2. If $\mathfrak{M}$ is a p.h.m., then from any point x of the cone K_0 such that $a_{t,0}(x) \neq 0$ $(t \in E)$, there issues an efficient trajectory.

Theorem 3. In order that a trajectory $\bar{\chi} = (\bar{x}_t)_{t\in E}$ of a p.h.m. $\mathfrak{M}$ be strongly efficient, it is necessary and sufficient that there exist a family $(f_t)_{t\in E}$ $(f_t \in K_t^*)$ possessing the following properties: 1) for any trajectory $\chi = (x_t)_{t\in E}$ of the model $\mathfrak{M}$, the function h, defined on E by the formula $h(t) = f_t(x_t)$, does not increase; 2) $h(t) \equiv f_t(\bar{x}_t) = 1$ for any $t \in E$. If the family $(f_t)_{t\in E}$ satisfies conditions 1) and 2), then for any t and $\tau > t$ $(\tau, t \in E)$

$$\max_{y\in a_{t,0}(\bar{x}_0)} f_t(y) = f_t(\bar{x}_t) = \min_{z\in\hat{a}_{\tau,t}(\bar{x}_\tau)} f_t(z). \qquad (1)$$

We shall give a characterization of weakly efficient trajectories. First we introduce the following definition. Let $\mathfrak{M}$ be a p.h.m., $t, \tau \in E$, $\tau > t$. For $f \in K_t^*$ put

$$a'_{\tau,t}(f) = \{g \in K_t^* \mid f(x) \geq \max_{y\in a_{\tau,t}(x)} g(y) \text{ for any } x \in K_t\}. \qquad (2)$$

* By the symbol K^* we denote the cone conjugate to K.

Theorem 4. Let $\overline{\chi} = (\overline{x}_t)_{t\in E}$ be a trajectory of the finite p.o.m. $\mathfrak{M}$. In order that there exist a family $(f_t)_{t\in E}$ $(f_t \in K_t^*,\ f_t \neq 0)$ such that: 1) for any trajectory $\chi = (x_t)_{t\in E}$ the function $h(t) = f_t(x_t)$ does not increase; 2) the function $\overline{h}(t) = f_t(\overline{x}_t)$ is constant, it is necessary and sufficient that there exist a functional f from the cone K_0^* possessing the following properties:

$$\alpha)\quad f(\overline{x}_0) = \min_{x \in a_{T,0}(x_T)} f(x); \qquad \beta)\quad a'_{T,0}(f) \neq \{0\} \quad (T = \sup E).$$

Corollary. Let the finite p.o.m. $\mathfrak{M}$ be such that $a'_{T,0}(f) \neq \{0\}$ for every $f \neq 0$. Then for every weakly efficient trajectory $\overline{\chi} = (\overline{x}_t)_{t\in E}$ of this model there exists a family $(f_t)_{t\in E}$ satisfying conditions 1) and 2) of Theorem 4.

4°. In this section an infinite-dimensional analogue of a p.o.m. is considered.* Let Q be a metric compactum. By the symbol $\Phi_{KR}(Q)$ we shall denote the space of all completely additive functions defined on the σ-algebra of Borel subsets of the compactum Q, endowed with the Kantorovich–Rubinstein norm ([8]). By the symbol $\mathrm{Lip}(Q)$ we shall denote the space of all functions defined on Q and satisfying there the Lipschitz condition ($\mathrm{Lip}(Q)$ is the space conjugate to $\Phi_{KR}(Q)$). By the symbol $\mathrm{Lip}^+(Q)$ we shall denote the cone of nonnegative functions from $\mathrm{Lip}(Q)$. Consider the model

$$\mathfrak{M} = \{E,\ (Q_t)_{t\in E},\ (\mathrm{Lip}(Q_t))_{t\in E},\ (\mathrm{Lip}^+(Q_t))_{t\in E},\ (a_{\tau,t})_{\tau,t\in E;\ \tau>t}\}. \tag{3}$$

Here, as above, E is a more than one-point set of nonnegative numbers containing zero; Q_t is a metric compactum ($t \in E$); $a_{\tau,t}$ is a superlinear mapping of the cone $\mathrm{Lip}^+(Q_t)$ into the collection of subsets of the cone $\mathrm{Lip}^+(Q_\tau)$ such that: 1) $a_{\tau,t}(u)$ is a compactum ($u \in \mathrm{Lip}^+(Q_t)$); 2) if $\varphi \in \Phi_{KR}(Q_\tau)$, $\varphi \geq 0$, then the functional q_φ:

$$q_\varphi(u) = \max_{v \in a_{\tau,t}(u)} \int_{Q_t} v\, d\varphi \qquad (u \in \mathrm{Lip}^+(Q_t))$$

is upper semicontinuous in $\sigma(\mathrm{Lip}(Q_t), \Phi_{KR}(Q_t))$. Moreover,

$$a_{t,t'} \circ a_{t',t''} = a_{t,t''} \qquad (t > t' > t'';\ t, t', t'' \in E).$$

A trajectory of model (3) will mean a family $\chi = (u_t)_{t\in E}$ such that: 1) $u_t \in \mathrm{Lip}^+(Q_t)$ ($t \in E$); 2) if $\tau, t \in E$, $\tau > t$, then $u_\tau \in a_{\tau,t}(u_t)$. Theorem 1 (on the existence of trajectories) is also valid for the model under consideration. A trajectory $\overline{\chi} = (\overline{u}_t)_{t\in E}$ will be called efficient if $\overline{u}_0(s) > 0$ ($s \in Q_0$) and, in addition, for every $t \in E$ there exists $\varphi_t \in \Phi_{KR}(Q_t)$ ($\varphi_t \geq 0,\ \varphi_t \neq 0$) such that

$$\int_{Q_t} \overline{u}_t\, d\varphi_t = \max_{v \in a_{\tau,t}(\overline{u}_t)} \int_{Q_t} v\, d\varphi_t.$$

(This definition, as is not difficult to verify, is equivalent to the definition of an efficient trajectory proceeding from an interior point in a p.o.m.)

Theorem 5. Let $u \in \mathrm{Lip}(Q_0)$, $u(s) > 0$ $(s \in Q_0)$. Then there exists an efficient trajectory of model (3) proceeding from u.

Theorem 6. Let $u \in \mathrm{Lip}(Q_0)$, $u(s) > 0$ $(s \in Q_0)$. In order that the trajectory $\overline{\chi} = (\overline{u}_t)_{t \in E}$, proceeding from u, be efficient, it is necessary and sufficient that there exist a family $(\varphi_t)_{t \in E}$ $(\varphi_t \in \Phi_{KR}(Q_t),\ \varphi_t \geq 0)$, possessing the following properties: 1) for any trajectory $\chi = (u_t)_{t \in E}$ of the model $\mathfrak{M}$ the function

$$h(t) = \int_{Q_t} u_t \, d\varphi_t$$

does not increase; 2)

$$\overline{h}(t) = \int_{Q_t} \overline{u}_t \, d\varphi_t = 1$$

for

* The importance of this analogue was pointed out to the author by G. Sh. Rubinstein.

any $t \in E$. If the family $((\varphi_t)_{t \in E})$ satisfies conditions 1) and 2), then for any t and $\tau > t$ $(\tau, t \in E)$

$$\max_{v \in a_{T,0}(\bar{u}_t)} \int_{\dot{Q}_t} v \, d\varphi_t = \int_{\dot{Q}_t} \bar{u}_t \, d\varphi_t = \min_{v \in \hat{a}_{\tau,t}(\bar{u}_t)} \int_{\dot{Q}_t} v \, d\varphi_t.$$

5°. The results presented above were obtained with the aid of the duality apparatus developed in (2). The following plays an important role here.

Theorem 7. Let $\mathfrak{M} = \{E,\ (X_t)_{t \in E},\ (K_t)_{t \in E} (a_{\tau,t})_{\tau,t \in E;\ \tau > t}\}$ be a p.o.m. Then the object $\mathfrak{M}' = \{E,\ (X_t^*)_{t \in E},\ (K_t^*)_{t \in E},\ (a_{\tau,t})_{\tau,t \in E;\ \tau > t}\}$ is also a p.o.m. (here $a_{\tau,t}$ is the mapping defined by formula (2)).

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Note: Figure translations are in progress. See original paper for figures.