ON SOME SIMPLEST GENERALIZATIONS OF LINEAR ELLIPTIC BOUNDARY-VALUE PROBLEMS

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Abstract

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In Euclidean n-dimensional point space x with Cartesian orthogonal coordinates $x_1, x_2, \dots, x_n$, consider a domain D, whose boundary S is an $(n-1)$-dimensional piecewise smooth Lyapunov surface. Denote by σ the part of S that is an $(n-1)$-dimensional open Lyapunov surface with parametric equation

$$x = x(t), \quad t = (t_1, t_2, \dots, t_{n-1}) \in \delta.$$

Let σ_0 be the diffeomorphic image $y = T(x)$ of σ, lying in the domain D, with parametric equation $y = y(t)$, $t \in \delta$.

In the domain D consider a uniformly elliptic linear differential operator of second order

$$E = A^i \partial^2 / \partial x_i \partial x_j + B^i \partial / \partial x_i + C$$

with real sufficiently smooth matrix coefficients of size $m \times m$.

The following boundary-value problem may be regarded as a natural generalization of the Dirichlet problem: find a regular (twice continuously differentiable) solution $u(x)$ in the domain D of the equation

$$Eu = f(x), \quad x \in D, \tag{1}$$

continuous in $\overline{D}$ and satisfying the conditions

$$u(x) = \varphi(x), \quad x \in S - \sigma; \tag{2}$$

$$u(y) = u(x), \quad y = T(x), \quad x(t) \in \sigma, \quad y(t) \in \sigma_0, \tag{3}$$

where $f(x)$ and $\varphi(x)$ are given real m-dimensional continuous vectors.

Below we restrict ourselves to the case where (1) is the two-dimensional Laplace equation $\Delta u = 0$ with independent variables x, y, the domain D coincides with the rectangle $-l < x < l,\ 0 < y < 1$, and σ and σ_0 are respectively the segments $x = l,\ 0 \le y \le 1$ and $x = 0,\ 0 \le y \le 1$.

Thus, one seeks a function $u(x, y)$ harmonic in the rectangle D, continuous in the closed rectangle $\overline{D}$, and satisfying the conditions

$$u(x, 0) = \varphi_1(x), \quad u(x, 1) = \varphi_2(x), \quad -l \le x \le l,$$

$$u(-l, y) = \varphi_3(y), \quad 0 \le y \le 1; \tag{4}$$

$$u(0, y) = u(l, y), \quad 0 \le y \le 1, \tag{5}$$

where φ_1, φ_2, and φ_3 are given continuous functions.

The uniqueness of the solution of problem (4)–(5) follows from the maximum principle for harmonic functions.

Indeed, taking condition (5) into account, on the basis of the indicated principle we conclude that a harmonic function satisfying conditions (4) $\cdots$

the function $u(x, y)$ can attain its extreme values in the closed rectangle $\overline{D}$ only on its left, upper, and lower sides. Consequently, the corresponding homogeneous problem (4)–(5) cannot have a nonzero solution, and thus the uniqueness of the solution of problem (4)–(5) is proved.

Let us denote the as yet unknown values of the desired solution $u(x, y)$ for $x = l,\ 0 \le y \le 1$ by $\varphi(y)$, i.e.,

$$\varphi(y) = u(l, y), \quad 0 \le y \le 1. \tag{6}$$

The solution of the Dirichlet problem with boundary conditions (4)–(5) is given by the well-known formula

$$u(x, y) = \int_{-l}^{l} K(x, y; \xi, 0)\varphi_1(\xi)\, d\xi + \int_0^1 K(x, y; l, \eta)\varphi(\eta)\, d\eta -$$

$$- \int_{-l}^{l} K(x, y; \xi, 1)\varphi_2(\xi)\, d\xi - \int_0^1 K(x, y; -l, \eta)\varphi_3(\eta)\, d\eta, \tag{7}$$

where the function $K(x, y; \xi, \eta)$ coincides with the derivative of the harmonic Green' s function $G(x, y; \xi, \eta)$ of the Dirichlet problem in the rectangle D with respect to the interior normal to its contour at the point (ξ, η).

By virtue of condition (5), from (7) we obtain the Fredholm integral equation of the second kind, equivalent to problem (4)–(5),

$$\varphi(y) - \int_0^1 K(0, y; l, \eta)\varphi(\eta)\, d\eta = \int_{-l}^{l} K(0, y; \xi, 0)\varphi_1(\xi)\, d\xi -$$

$$- \int_{-l}^{l} K(0, y; \xi, 1)\varphi_2(\xi)\, d\xi - \int_0^1 K(0, y; -l, \eta)\varphi_3(\eta)\, d\eta, \tag{8}$$

whose kernel $K(0, y; \xi, 0)$ is an analytic function of the variables y, η, becoming infinite of order $1/2$ for $\eta = 0$ and $\eta = 1$.

Since problem (4)–(5) is equivalent to the integral equation (8), the solvability of the latter and, consequently, the existence of a solution of problem (4)–(5) follow from the uniqueness property of this solution proved above.

The boundary-value problem (4)–(5) is studied analogously in the case when the third of conditions (4) is replaced by the condition $u(-l, y) = u(0, y)$.

In particular, the solution of the problem of finding a harmonic function $u(x, y)$ in the rectangle D, continuous in $\overline{D}$ and satisfying the conditions

$$u(x, 0) = \varphi_1(x), \quad u(x, 1) = \varphi_2(x), \quad 0 \le x \le l,$$

$$u(x, y) = u(x + l, y), \quad -l \le x \le 0, \quad 0 \le y \le 1,$$

where φ_1 and φ_2 are prescribed continuous functions, is given by the formula

$$u(x, y) = \frac{1}{2} \sum_{k=-\infty}^{\infty} \int_0^l \left[\frac{\varphi_1(t)}{\operatorname{ch} \pi(t + kl - x) - \cos \pi y} + \frac{\varphi_2(t)}{\operatorname{ch} \pi(t + kl - x) + \cos \pi y} \right] \sin \pi y\, dt.$$

The method applied above is suitable for proving the existence and uniqueness of a solution of problem (1)–(2)–(3) in those cases when, for solutions of equation (1) in the domain D, the extremum principle holds. With a slight modification, the same method leads to the goal in all those cases when equation (1) in the domain D has a fundamental solution.

When the boundary condition (2) is replaced by a general linear boundary condition (for example, the condition of the Poincaré problem), the investigation of the resulting problem encounters the same difficulties as arise in the case where the support of the analogous condition is the entire boundary S of the domain D.

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Note: Figure translations are in progress. See original paper for figures.