ON A TRANSFORMATION OF CONVEX FUNCTIONS AND THE DUALITY OF THE \(\beta\)- AND \(\delta\)-CHARACTERISTICS OF A \(B\)-SPACE

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Abstract

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MATHEMATICS

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ON A TRANSFORMATION OF CONVEX FUNCTIONS AND THE DUALITY OF THE β- AND δ-CHARACTERISTICS OF A B-SPACE

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1. Let $\mathfrak{B}$ be some family of subspaces of the Banach space B. Consider the following averages of real bounded functions $f(x)$ for $x \in S(B) = \{x \in B : \|x\| = 1\}$:

$$\beta[f;\mathfrak{B}] = \sup_{E\in\mathfrak{B}} \inf_{x\in S(E)} f(x); \qquad \delta[f;\mathfrak{B}] = \inf_{E\in\mathfrak{B}} \sup_{x\in S(E)} f(x). \tag{1}$$

In the case when $\mathfrak{B}$ is the family of all subspaces of finite defect, we shall write $\beta[f,B] = \beta f$ and $\delta[f,B] = \delta f$. In papers ([1,2]) the averages (1) were studied for the special function $f(\varepsilon;x,y) = \|x+\varepsilon y\| - 1$ with respect to the variable $y \in S(B)$ (here $\beta_y[f,\mathfrak{B}]$ is denoted by $\beta^0(\varepsilon;x,\mathfrak{B})$ and $\delta_y[f,\mathfrak{B}] = \delta^0(\varepsilon;x,\mathfrak{B})$) and the subsequent averaging of the obtained functions β^0 and δ^0 with respect to the variable $x \in S(B)$: $\beta\beta(\varepsilon;B) = \beta[\beta^0(\varepsilon;x,B),B]$, $\delta\delta(\varepsilon;B) = \delta[\delta^0(\varepsilon;x,B),B]$.

In the present paper (Sec. 4) the relation between the β- and δ-characteristics of a space and of its conjugate is studied. In this connection there arises a functional transformation (see Sec. 2) close to the Legendre transform: $\psi(\eta) = \sup\{\xi\eta - \varphi(\xi) : \xi \geq 0\} = L_0(\varphi)$. Preliminarily, in Sec. 3 special minimal systems are considered, and in Sec. 5—some results on complementability.

In what follows, $E(M)$ denotes the closed linear span of M; $B_1 \simeq B_2$ means an isomorphism of the spaces B_1 and B_2; $\{x_k\}_1^\infty \simeq \{y_k\}_1^\infty$ means that the sequences $X = \{x_k\}_1^\infty$ and $Y = \{y_k\}_1^\infty$ are isomorphic, i.e. the bounded linear operators $T : E(X) \to E(Y)$ and T^{-1} exist, where $\{Tx_k = y_k\}_1^\infty$. A sequence $\{x_k\}_1^\infty \subset B$ is called minimal if there exists a conjugate system of functionals $\{f_k\}_1^\infty \subset B^*$, $f_k(x_j) = \delta_{kj}$. A basis $\{x_k\}_1^\infty \subset B$ is called orthogonal if for any m and n $(m > n)$

$$\left\|\sum_n^\infty a_k x_k\right\| \geq \left\|\sum_m^\infty a_k x_k\right\|.$$

2. On a transformation of convex functions.

Theorem 1. The transformation

$$\psi(\xi) = \mathcal{L}_1[\varphi(\eta)] = \sup_{\eta \geq 0} \frac{\xi\eta - \varphi(\eta)}{1 + \varphi(\eta)}, \tag{2}$$

where $1 + \varphi(\eta) > 0$ for $\eta \geq 0$, has the following properties:

a) for $\xi \geq 0$, $\psi(\xi)$ is a convex nondecreasing function, $1 + \psi(\xi) \geq 0$, and the function $(\psi(\xi) + 1)/\xi$ does not increase;

b) inversion formula: in the class of functions satisfying property a), there exists a unique function $\varphi(\eta) = \mathcal{L}_1[\psi(\xi)]$ for which (2) is fulfilled;

c) for an arbitrary function $\varphi(\eta)$, the inverse transform $\mathcal{L}_1[\psi] = \mathcal{L}_1[\mathcal{L}_1(\varphi)]$ gives the function

$$\widehat{\varphi}(\eta) = \sup\{r(\eta) : r(\eta) \leq \varphi(\eta) \text{ and } r(\eta) \text{ satisfies property a)}\}.$$

Examples. For $p \geq 1$,

$$\mathcal{L}_1[(1 + \eta^p)^{1/p} - 1] = (1 + \xi^q)^{1/q} - 1,$$

where

$$p^{-1} + q^{-1} = 1.$$

3. Normalizing families of subspaces. A family $\mathfrak{B} = \{E_\alpha\}$ of subspaces $E_\alpha \subset B$ will be called c-normalizing

$(c > 0)$, if the seminorm

$$\|x\|_{\mathfrak{B}} = \sup_{E \in \mathfrak{B}} \inf_{y \in E} \|x + y\| \tag{3}$$

is equivalent to the original norm of the B-space and $\|x\| \geq \|x\|_{\mathfrak{B}} \geq c\|x\|$ for all $x \in B$. For us the case of a 1-norming family is especially important; for it $\|x\|_{\mathfrak{B}} = \|x\|$. Let us note that (3) is conveniently understood in the following way: let $P_E x$ denote the image of x in the quotient space B/E, whose norm we denote by $\|\cdot\|_E$; then

$$\|x\|_{\mathfrak{B}} = \sup_{E \in \mathfrak{B}} \|P_E x\|_E. \tag{3'}$$

From $(3')$ it is clear that the notion of a norming family $\mathfrak{B}$ contains as a special case the notion of a norming family of functionals, which was widely used by M. I. Kadets, and is its generalization.

Remark 1. Let $X = \{x_k\}_1^\infty$ be a minimal system in B with a 1-norming adjoint system $\{f_k\}_1^\infty \subset B^*$, i.e. $F = E(\{f_k\})$ is a 1-norming subspace in B^*: $\sup\{|f(x)| : f \in F\} = \|x\|$ for any $x \in B$. Denote

$$\mathfrak{B}(X) = \{E(\{x_k\}_{k=n+1}^\infty)\}_1^\infty.$$

Then $\|x\|_{\mathfrak{B}} = \|x\|$ for all $x \in B$.

As noted in (3), every separable B-space has a complete minimal system with a 1-norming adjoint.

Theorem 2. Let $X = \{x_k\}_1^\infty$ be a stretching basis in B, i.e. the adjoint system $X^* = \{f_k\}_1^\infty$ is complete in B^*. Then for any $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that, for every N, one can choose N_1 in such a way that for all $y \in E^N = E(\{x_k\}_{k=N_1+1}^\infty)$

$$\|y\| \le (1+\varepsilon) f(y)$$

for some $f \in S(E(\{f_k\}_{k=1}^{n_0}, \{f_k\}_{k=N+1}^\infty))$.

Proposition 1. Let $\{x_k\}_1^\infty \subset B$ be a complete minimal sequence with a 1-norming adjoint $\{f_k\}_1^\infty \subset B^*$. Then in B one can introduce an equivalent norm $\|x\|_1$: $\|x\| \le \|x\|_1 \le 2\|x\|$, so that for any n and $\varepsilon > 0$ there is an $N = N(\varepsilon, n)$ for which, for every $y \in E^N =$

$$= E(\{x_k\}_{k=N+1}^\infty) \quad \sup\{f(y) : f \in E(\{f_k\}_{k=n}^\infty),\ \|f\|_1 = 1\} \ge (1-\varepsilon)\|y\|_1.$$

4. The relation between the β- and δ-functions of a B-space and its adjoint

The results of this section essentially use the properties of minimal systems indicated above.

Theorem 3. Let $X = \{x_k\}_1^\infty$ be an orthogonal basis in B and let $X^* = \{f_k\}_1^\infty$ be the biorthogonal functionals in B^* (i.e. $f_k(x_j) = \delta_{kj}$), and suppose $E(X^*) = B^*$. Then

$$\delta\delta(\varepsilon; B) = \mathcal{L}_1[\beta\beta(\eta; B^*)] = \sup_{\eta \ge 0} \frac{\xi\eta - \beta\beta(\eta, B^*)}{1 + \beta\beta(\eta, B^*)}. \tag{4}$$

The conditions under which Theorem 3 is true can be weakened considerably; however, because of their cumbersome form we do not dwell on this.

Denote

$$\sup\{f(\varepsilon; x) : x \in S(B)\} = sf(\varepsilon; B)$$

and

$$\inf\{f(\varepsilon; x) : x \in S(B)\} = if(\varepsilon; B).$$

Theorem 4. Let B be a reflexive separable space. Then

$$s\delta(\varepsilon; B) \leq \mathcal{L}_1[i\beta(\eta; B^*)] = \sup_{\eta \geq 0} \frac{\xi\eta - i\beta(\eta, B^*)}{1 + i\beta(\eta, B^*)} \leq s\delta(2\varepsilon; B).$$

Let us note that if in B one introduces the equivalent norm indicated in Proposition 1, then the functions $s\delta(\varepsilon)$ and $i\beta(\eta)$ will be related by equality (4).

We also note the following proposition, based on Theorem 2.

Proposition 2. Let there exist a stretching basis in B. Then

$$s\delta(\varepsilon; B) \leq \mathcal{L}_1[i\beta(\eta, B^*)]. \tag{5}$$

Relations (4) and (5) connect the functions $\beta(\varepsilon)$ and $\delta(\varepsilon)$. In order to express the function $\beta(\varepsilon)$ in terms of $\delta(\varepsilon)$, we shall need the properties of the transformation (2) indicated in Theorem 1. The function $\beta\beta(\varepsilon; B)$ is nondecreasing

and $[\beta\beta(\varepsilon; B) + 1]/\varepsilon$ does not increase*, but, generally speaking, one cannot assert that the function $\beta\beta(\varepsilon; B)$ is convex; therefore formulas (4) and (5) should be treated by applying part c) of Theorem 1. Let us illustrate this by the following example.

Corollary 1. *Let B^* be separable and, for some $c > 0$,*

$$\delta\delta(\varepsilon; B) \geq c\varepsilon.$$

Then $\beta\beta(\eta; B^*) = 0$ for $\eta \leq c$. If

$$\delta\delta(\varepsilon; B^*) \geq c\varepsilon,$$

then $\beta\beta(\eta; B) = 0$ for $\eta \leq c$.

5. On one criterion for complemented subspaces

A basic sequence $\{x_k\}_1^\infty$ is called boundedly complete (see, for example, ([6]), p. 119) if from

$$\left\|\sum_1^n a_{kx} k\right\| < M$$

for all n there follows the convergence of the series

$$\sum_1^\infty a_{kx} k.$$

Let us note that in a reflexive space every basic sequence is boundedly complete.

Theorem 5. *Let $\{x_k\}_1^\infty$ be a boundedly complete basic sequence in B, $E(\{x_k\}_1^\infty) = B_1 \subset B$. For the complementability of B_1 in B it is necessary and sufficient that there exist a biorthogonal system $\{f_k\}_1^\infty \subset B^*$ ($f_k(x_j) = \delta_{kj}$) such that $B_2 = E(\{f_k\}_1^\infty)$ and B_1 are mutually norming subspaces.*

Remark 2. If $\{f_k\}_1^\infty$ is a basic sequence or if B_1 is a reflexive subspace, then it is sufficient to require that B_2 be norming over B_1.

Corollary 2. *Denote by $\{\varphi_k\}_1^\infty \subset B_1^*$ the biorthogonal system to $\{x_k\}_1^\infty$ ($\varphi_k(x_j) = \delta_{kj}$). If $\{\varphi_k\}_1^\infty \simeq \{f_k\}_1^\infty$, then B_1 is complemented in B.*

It is known ([4]) that for $p \geq 2$ every subspace $E \subset L_p$ and $E \simeq l_2$ is complemented in L_p. For $4/3 < p < 2$ this is not so (see ([5])); however, from Corollary 2 we easily obtain:

Proposition 3. *If $l_2 \simeq E \subset L_p$ $(1 < p < 2)$, then there exists an infinite-dimensional subspace $E_1 \subset E$, and E_1 is complemented in L_p.*

Let us also note that a consequence of Theorem 5 is Alaoglu' s theorem (see ([6]), p. 120): if B has a boundedly complete basis, then B is isomorphic to a conjugate space.

6. Some applications of the duality theorems

It is easy to see that for James' space (for the definition see ([6]), p. 123) J,

$$\beta^0(\varepsilon; x, J) = \delta^0(\varepsilon; x, J) = \sqrt{1+\varepsilon^2} - 1.$$

It follows from Theorem 3 that in the conjugate space J^*,

$$\beta^0(\varepsilon; x, J^*) = \delta^0(\varepsilon; x, J^*) = \sqrt{1+\varepsilon^2} - 1.$$

But then (see ([1])) any subspaces $E_1 \subset J$ and $E_2 \subset J^*$ ($\dim E_i = \infty$) contain $E_{1,1} \subset E_1$ and $E_{2,1} \subset E_2$, $E_{1,1} \simeq l_2 \simeq E_{2,1}$. Analogous reasoning with sequences and the use of Corollary 2 shows that: every $E \subset J$ ($\dim E = \infty$) contains $E_1 \subset E$ ($\dim E_1 = \infty$), and E_1 is complemented in J.

At the same time I think that the following stronger proposition holds:

Hypothesis. Every subspace $E \subset J$ is complemented in J.

We now give two results in which the use of Corollary 1 makes it possible to strengthen the corresponding theorems from ([2]).

Theorem 6. *If B has an unconditional basis and*

$$\delta^0(\varepsilon; x, B) \geq c\varepsilon \quad (c > 0),$$

then there exists $B_1 \subset B$, and $B_1 \simeq l_1$.

Theorem 7. *If*

$$\delta^0(\varepsilon; x, B) \geq c\varepsilon \quad (c > 0),$$

then B is not reflexive and in B there does not exist a shrinking basis $\{x_k\}_1^\infty$ (i.e., one such that $E(\{f_k\}_1^\infty) = B^*$, where $f_k(x_j) = \delta_{kj}$).

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* The same is true also for the function $i\beta(\varepsilon; B)$.

Note: Figure translations are in progress. See original paper for figures.