BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF SECOND-ORDER EQUATIONS OF COMPOSITE TYPE POSSESSING ONE MULTIPLE FAMILY OF REAL CHARACTERISTICS

MATHEMATICS

1969

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Abstract

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UDC 517.944

MATHEMATICS

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BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF SECOND-ORDER EQUATIONS OF COMPOSITE TYPE POSSESSING ONE MULTIPLE FAMILY OF REAL CHARACTERISTICS

(Presented by Academician I. N. Vekua on 29 X 1968)

In the author' s preceding paper ([1]), a formula was obtained indicating the structure of regular solutions of the following system of equations with second-order partial derivatives of composite type:

$$D(u) \equiv u_{\bar{z}y} - q(z)u_{zy} + A(z)u_x + B(z)\bar{u}_x + \\ +C(z)u_y + D(z)\bar{u}_y + E(z)u + F(z)\bar{u} = G(z), \\ |q(z)| \leq \text{const} < 1. \tag{1}$$

There it was also pointed out that a system of equations of the general form

$$A^0(z)u_{xx} + 2B^0(z)u_{xy} + C^0(z)u_{yy} + A(z)u_x + \\ +B(z)\bar{u}_x + C(z)u_y + D(z)\bar{u}_y + E(z)u + F(z)\bar{u} = G(z)$$

can be reduced, in a neighborhood of any point of the domain, to a system of the form (1), provided that its characteristic polynomial

$$p(\lambda) = \left(A^0(z)\lambda^2 + 2B^0(z)\lambda + C^0(z)\right)\left(\overline{A^0(z)}\lambda^2 + 2\overline{B^0(z)}\lambda + \overline{C^0(z)}\right)$$

possesses, throughout the entire domain under consideration, both real and imaginary roots.

In the present paper mutually adjoint boundary value problems for system (1) and for the system adjoint to it will be posed and investigated.

1°. Let G be a simply connected bounded domain in the plane $z = x + iy$, possessing the following properties: the boundary Γ of the domain G is a closed curve of class C^1_ν, and is such that the straight lines $x = a$, $x = b$ $(a < b)$ touch it respectively at the points M and N; the straight lines $x = c$, for $a < c < b$, intersect Γ in exactly two points, while the straight lines $x = c$ for $c < a$ and $c > b$ have no common points with Γ. The points M and N divide the curve Γ into two arcs γ_1 and $\gamma_2 = \Gamma - \gamma_1$, and these arcs, when approaching the points M and N from above and from below, have the same order.

Problem I. Find, in the domain G, regular solutions of equation (1), continuous in the closed domain $\overline{G} = G+\Gamma$ together with the partial derivative with respect to y, and satisfying the conditions:

$$\mathrm{Re}[i\Delta(t_0)u_y(t_0)] = h^{(0)}(t_0), \qquad t_0 \in \gamma_1; \tag{2}$$

$$u(t_0) = d(t_0)\overline{u_y(t_0)} + h^{(1)}(t_0), \qquad t_0 \in \gamma_1; \tag{3}$$

$$\mathrm{Re}[a_0(t_0)u(t_0) + b_0(t_0)u_y(t_0)] = h_0(t_0), \qquad t_0 \in \gamma_2, \tag{4}$$

where $u_y(t_0)$ denotes the boundary value of the function $u_y(z)$ from inside the domain G; $\Delta(t_0)$, $d(t_0)$, $h^{(0)}(t_0)$, $h^{(1)}(t_0)$ are functions prescribed on the arc γ_1, continuous in the Hölder sense; $a_0(t_0)$, $b_0(t_0)$, $h_0(t_0)$ are functions prescribed on the arc γ_2 and continuous there in the Hölder sense.

In what follows we shall assume that the following conditions are satisfied: on the arc $\bar{\gamma}_1$ the inequality

$$\Delta(t_0) \neq 0, \qquad t_0 \in \bar{\gamma}_1, \tag{5}$$

is fulfilled, and on the arc $\bar{\gamma}_2$ the inequality

$$b_0(t_0) \neq 0, \qquad t_0 \in \bar{\gamma}_2, \tag{6}$$

and, moreover, at the points $P = M, N$ the equalities

$$i\Delta(P) = b_0(P); \qquad a_0(P) = d(P) = 0; \qquad h^{(0)}(P) = h_0(P). \tag{7}$$

Let us note that Problem I may at first glance seem to be a problem of a rather special character. However, this problem is so general that, in view of (2)–(4), boundary conditions of the general form (cf. (4))

$$\mathrm{Re}\,[a_0(t_0)u(t_0) + b_0(t_0)u_y(t_0)] = h_0(t_0), \qquad t_0 \in \Gamma,$$

$$\mathrm{Re}\,[a_k(t_0)u(t_0) + b_k(t_0)u_y(t_0)] = h_k(t_0), \qquad t_0 \in \gamma_1,\ k = 1, 2,$$

can be reduced to it, provided only that $b_0(t_0) \neq 0$ for $t_0 \in \bar{\gamma}_2$ and $\Delta(t_0) = 2i(b_0(t_0)a_{12}(t_0) - b_1(t_0)a_{02}(t_0) + b_2(t_0)a_{01}(t_0)) \neq 0$ for $t_0 \in \bar{\gamma}_1$, where $a_{ij}(t_0) = \mathrm{Im}(a_i(t_0)a_j(t_0))$. If $G(z) = h^{(0)}(t_0) = h^1(t_0) = h_0(t_0) \equiv 0$, then the formulated problem will be called the homogeneous problem I_0. The following homogeneous problem will be called adjoint to Problem I:

Problem I_0^*. Find solutions regular in G of the equation adjoint to (1),

$$D^*(u^*) \equiv u^*_{zy} - (qu^*)_{zy} - (Au^*)_x - (\bar{B}\bar{u}^*)_x$$

$$-(Cu^*)_y - (\bar{D}\bar{u}^*)_y + E(z)u^* + \bar{F}(z)\bar{u}^* = 0, \qquad (1^*)$$

continuous in the closed domain $\bar{G}$ together with the first derivatives and satisfying the conditions:

$$\mathrm{Re}\left[\frac{\theta(s_0)}{2i\Delta(t_0)}u^*(t_0) + \frac{\overline{d(t_0)}}{\Delta(t_0)}\overline{B^*(t_0, u^*)}\right] = 0, \qquad t_0 \in \gamma_1; \qquad (8)$$

$$\mathrm{Re}\left[\frac{\theta(s_0)}{b_0(t_0)}u^*(t_0)\right] = 0, \qquad t_0 \in \gamma_2; \qquad (9)$$

$$\frac{a_0(t_0)\theta(s_0)}{2ib_0(t_0)}u^*(t_0) - B^*(t_0, u^*) = 0, \qquad t_0 \in \gamma_2, \qquad (10)$$

where $t = t(s)$ is the parametric equation of the contour Γ; $\theta(s) = t'(s) + q(t)\overline{t'(s)}$,

$$B^*(t, u^*) \equiv \xi'(s)\left[u^*_{\bar{z}}(t) - (qu^*)_z(t) - C(t)u^*(t) - \overline{D(t)}\,\overline{u^*(t)}\right]$$

$$+\eta'(s)\left[A(t)u^*(t) + \overline{B(t)}\,\overline{u^*(t)}\right], \qquad (11)$$

where $u^*_{\bar{z}}(t)$, $(qu^*)_z(t)$ are the boundary values of the functions $u^*_{\bar{z}}(z)$ and $(q(z)u^*(z))_z$ at the point $t \in \Gamma$.

The connection between Problems I and I_0^* is established by

Theorem 1. *For Problem I to be solvable it is necessary that its right-hand sides satisfy the equality*

$$\mathrm{Re}\iint_G G(z)u^*(z)\,dG_z + \mathrm{Re}\int_{\gamma_1} h^{(1)}(t)B^*(t, u^*)\,ds +$$

$$+\int_{\gamma_1} \frac{h^{(0)}(t)}{i\Delta(t)} \left\{ \frac{\theta(s)}{2i} u^*(t) + \overline{d(t)}\,\overline{B^*(t,u^*)} \right\} ds +$$

$$+\frac{1}{2i}\int_{\gamma_2} h_0(t) \frac{\theta(s)}{b_0(t)} u^*(t)\, ds = 0, \tag{12}$$

where $u^*(t)$ is any solution of the problem I_0^*.

Proof. For any two complex-valued functions $u(z)$ and $u^*(z)$, twice differentiable in G and continuous in $\overline{G}$ together with their first derivatives, the following obvious identity holds:

$$\begin{aligned} \operatorname{Re}\left[u^*(z)D(u) - u(z)D^*(u^*)\right] \equiv \operatorname{Re}\{&(u^*(z)u_y)_{\bar z} - (q(z)u^*(z)u_y)_z \\ &- ([u^*_{\bar z} - (qu^*)_z - C(z)u^*(z) - \overline{D(z)}\,\overline{u^*(z)}]u(z))_y \\ &+ ([A(z)u^*(z) + \overline{B(z)}\,\overline{u^*(z)}]u(z))_x\}. \end{aligned} \tag{13}$$

Integrating identity (13) over the domain G and using Green' s formula, we obtain the equality

$$\begin{aligned} \operatorname{Re}\iint_G [u^*(z)D(u) - u(z)D^*(u^*)]\, dG_z &= \operatorname{Re}\int_\Gamma \left[B^*(t,u^*(t))u(t) + \frac{\theta(s)}{2i}u^*(t)u_y(t)\right] ds \\ &= \operatorname{Re}\int_{\gamma_1} \left[\overline{B^*(t,u^*)u(t)} + \frac{\theta(s)}{2i}u^*(t)u_y(t)\right] ds \\ &+ \operatorname{Re}\int_{\gamma_2} \left[B^*(t,u^*)u(t) + \frac{\theta(s)}{2i}u^*(t)u_y(t)\right] ds. \end{aligned} \tag{14}$$

If we now suppose that problem I is solvable and that the function $u(z)$ is its solution, then $D(u) \equiv G(z)$, and, by virtue of the boundary conditions of problem I,

$$u(t) = d(t)u_y(t) + h^{(1)}(t), \qquad i\Delta(t)u_y(t) = i\chi_1(t) + h^{(0)}(t) \quad \text{for } t \in \gamma_1,$$

$$b_0(t)u_y(t) + a_0(t)u(t) = i\chi_2(t) + h_0(t), \qquad t \in \gamma_2,$$

where $\chi_1(t)$, $\chi_2(t)$ are real functions defined respectively on the arcs γ_1 and γ_2. Therefore equality (14), for $u(z)$—the solution of problem I—takes the form:

$$\operatorname{Re}\iint_G [u^*(z)G(z) - u(z)D^*(u^*)]\,dG_z = \operatorname{Re}\int_{\gamma_1} h^{(1)}(t)B^*(t,u^*)\,ds$$
$$+\int_{\gamma_1} h^{(0)}(t)\operatorname{Re}\frac{1}{i}\left[\frac{\theta(s)}{2i\Delta(t)}u^*(t) + \frac{\overline{d(t)}}{\Delta(t)}B^*(t,u^*)\right]ds$$
$$+\int_{\gamma_1} \chi_1(t)\operatorname{Re}\left[\frac{\theta(s)}{2i\Delta(t)}u^*(t) + \frac{\overline{d(t)}}{\Delta(t)}B^*(t,u^*)\right]ds$$
$$+\int_{\gamma_2} h_0(t)\operatorname{Re}\left(\frac{\theta(s)}{2ib_0(t)}u^*(t)\right)ds$$
$$+\frac{1}{2}\int_{\gamma_2} \chi_2(t)\operatorname{Re}\left(\frac{\theta(s)}{b_0(t)}u^*(t)\right)ds$$
$$+\operatorname{Re}\int_{\gamma_2}\left[B^*(t,u^*) - \frac{a_0(t)\theta(s)}{2ib_0(t)}u^*(t)\right]u(t)\,ds. \tag{15}$$

Assuming now that $u^*(z)$ is a solution of problem I_0^*, by virtue of equation (1) and conditions (8)–(10), we see that equality (15) reduces to condition (12), as was required to prove.

2°. For the study of problems I and I_0^* I shall apply the method of singular integro-functional equations proposed by me in (2–4), from which, in particular, it will follow that conditions (12) for the solvability of problem I are not only necessary but also sufficient.

In (1) it was proved that all regular solutions of equation (1) in the domain G are representable in the form:

$$u(z) = u^0(z) + \iint_G \{R_1(\zeta,z)u^0(\zeta) + R_2(\zeta,z)\overline{u^0(\zeta)}\}\,d\sigma_\zeta + \sum_{k=1}^{N} C_k u_k(z), \tag{16}$$

where

$$u^0(z) = \omega(x) + \int_{\sigma_2(x)}^{y} \varphi(x+i\sigma)\,d\sigma + \iint_G \{K_1^0(\zeta,z)G(\zeta) + K_0^2(\zeta,x)\overline{G(\zeta)}\}\,dG_\zeta,$$

$\omega(x)$ is an arbitrary complex-valued function; $\varphi(z)$ is an arbitrary regular solution of the uniformly elliptic equation

$$\varphi_{\bar z} - q(z)\varphi_z + C(z)\varphi + D(z)\overline{\varphi} = 0, \tag{17}$$

satisfying, in addition, a finite number of equalities of the type of the vanishing of certain functionals of $\omega(x)$ and $\varphi(z)$. Using the representation (16) and the boundary condition (3), for determining the function $\omega(x)$ we obtain the following quasi-Fredholm singular integral equation on the interval $[a,b]$:

$$A_1(\xi_0)\omega(\xi_0)+\frac{B_1(\xi_0)}{\pi i}\int_a^b\frac{\omega(\xi)\,d\xi}{\xi-\xi_0}+A_2(\xi_0)\overline{\omega(\xi_0)}+\frac{B_2(\xi_0)}{\pi i}\int_a^b\frac{\overline{\omega(\xi)}\,d\xi}{\xi-\xi_0}+$$

$$+T(\omega)=d(t_0)\overline{\varphi(t_0)}+h^{(1)}(t_0)+\sum_{k=1}^{N}C_kh_k(t_0),\quad t_0=\xi_0+i\sigma_1(\xi_0)\in\gamma_1,$$

where $A_1(\xi_0)\pm B_1(\xi_0)\neq 0$.

Solving equation (18), we express the function $\omega(\xi_0)$ in terms of the function $\varphi(t_0)$ and the constants C_k. Then, representing the function $\varphi(z)$ in the form of a contour integral with real density $\mu(t)$, we reduce conditions (2) and (4) to a singular integro-functional equation for the function $\mu(t)$ (see (2-4)).

For the equivalence of the obtained singular integro-functional equation and problem I, it is necessary and sufficient that the $N+n+1$ real constants entering the right-hand side of the integro-functional equation satisfy an algebraic system of $N+n$ equations. Arguing further in the same way as in (2-4), we become convinced of the validity of the following theorem:

Theorem 2. Let $a^*(t)=i\Delta(t)$ for $t\in\gamma_1$, and $a^*(t)=a^0(t)$ for $t\in\gamma_2$. Then, under conditions (5)-(7), for the solvability of problem I it is necessary and sufficient that conditions of the form (12) be fulfilled; moreover, the corresponding homogeneous problems I_0 and I_0^* can have only a finite number of linearly independent solutions, and for the index of problem I the formula holds

$$\mathrm{Ind}(I)=1+\frac{1}{\pi}\{\arg\overline{a^*(t)}\}_\Gamma.$$

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Received
26 X 1968

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Note: Figure translations are in progress. See original paper for figures.