ON MAPPINGS THAT DO NOT LOWER DIMENSION

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 513.83+519.54

MATHEMATICS

V. V. FEDORCHUK

ON MAPPINGS THAT DO NOT LOWER DIMENSION

(Presented by Academician P. S. Aleksandrov on 2 VII 1968)

The classical Hurewicz formula shows that under a closed mapping of one metric space with a countable base onto another, the dimension cannot decrease by more than the number m, which is the dimension of the mapping*. E. G. Sklyarenko, by means of sheaf theory, proved ([1]) Hurewicz's formula for a closed mapping of a paracompact space onto a paracompact space, and B. A. Pasynkov generalized ([2]) E. G. Sklyarenko' s result to the case of a closed mapping of a normal space onto a paracompact space.

The simplest examples of perfect mappings (projections of a product onto one of the factors) show that the property of a perfect mapping to lower the dimension of a space is natural. At the same time, the property of a mapping not to lower dimension at first sight seems pathological, except, of course, for those cases when the dimension of the image space is not greater than one. All the more interesting is the fact that a sufficiently broad class of mappings has the property of not lowering dimension.

Definition. A continuous mapping $f : X \to Y$ is called **strongly closed** if, for any point $y \in Y$ and any finite cover $\{U_i \mid i = 1, \dots, j\}$ of its preimage $f^{-1}y$ by sets open in X, the set

$$\{y\} \cup \left(\bigcup_{i=1}^{j} f^{\#} U_i \right)$$

is open in Y**.

Every strongly closed mapping is closed. To verify this it suffices to consider a cover of the preimage consisting of a single element. The following theorem, which is a partial strengthening of the above-mentioned result of B. A. Pasynkov, shows that a strongly closed mapping whose dimension is less than the dimension of the image space does not lower dimension.

Theorem. *Let $f : X \to Y$ be a strongly closed mapping of a normal space X onto a paracompact space V. If $\dim V \le n$, $\dim f \le m$, then* $\dim X \le$

$\max\{m, n\}$.

Proof. Let $\omega = \{O_p,\ p = 1, \dots, q\}$ be an arbitrary finite cover*** of the space X. We shall show that into the cover ω one can inscribe a cover of multiplicity $\leq l + 1$, where $l = \max\{m, n\}$. Let y be an arbitrary point of the space Y. There exists a finite system $\alpha_y = \{A_1^y, \dots, A_{r(y)}^y\}$ of sets open in the space X, which is a cover of multiplicity $\leq l + 1$ of the set $f^{-1}y$, inscribed in the cover ω of the space X. The existence of such a system α_y follows from the following well-known proposition.

Lemma 1. *Let F be a closed subset of a normal space X, and let $\dim F \leq n$. Then for every finite cover ω of the space–*

* The dimension of a mapping $f : X \to Y$ is defined as follows:

$$\dim f = \sup_{y \in Y} \dim f^{-1} y.$$

** By $f^{\#} A$ is denoted the small image of the set A.

*** All covers are assumed to be open.

of the space X there exists a finite system α of multiplicity $\leq n + 1$ of open subsets of X, which is a cover of the set F inscribed in the cover ω.

Let us now consider the set

$$U_y = \{y\} \cup \left(\bigcup_{s=1}^{r(y)} f^{\#} A_s^y \right).$$

Since the mapping f is strongly closed, the set U_y is a neighborhood of the point y. Into the cover $\{U_y \mid y \in Y\}$ of the paracompact space Y one can inscribe a locally finite cover

$$\beta = \{B_\gamma \mid \gamma \in \Gamma\}.$$

For each element B_γ of the cover β we mark one point y for which $U_y \supset B_\gamma$; denote it by y_γ. Now, for each $\gamma \in \Gamma$, put

$$\tilde{\alpha}_\gamma = \{\widetilde{A}_1^\gamma, \dots, \widetilde{A}_{r(\gamma)}^\gamma\},$$

where

$$\widetilde{A}_s^\gamma = (f^{-1} B_\gamma) \cap \{(A_s^{y_\gamma} \cap f^{-1} y_\gamma) \cup f^{-1} f^{\#} A_s^{y_\gamma}\}.$$

Before dwelling on the properties of the system $\tilde{\alpha}_\gamma$, we prove one auxiliary assertion.

Lemma 2. Let $f : X \to Y$ be a strongly closed mapping of a regular space X onto a space Y, and let U be an open subset of the space X. Then for any point $y \in Y$ the set

$$U^y = (U \cap f^{-1} y) \cup f^{-1} f^{\#} U$$

is open.

Proof. We must show that every point $x \in U \cap f^{-1}y$ has a neighborhood contained in the set U^y. Since the space X is regular, there exists a neighborhood W of the point x such that $[W] \subset U$. Consider the binary cover

$$\{U,\ V = X \setminus [W]\}$$

of the space X. Since the mapping f is strongly closed, the set

$$H = \{y\} \cup (f^{\#}U \cup U f^{\#} V)$$

is a neighborhood of the point y. Then the set $W \cap f^{-1}H$ is a neighborhood of the point $x \in f^{-1}y$. The set $W \cap f^{-1}H$ consists of three subsets

$$W \cap f^{-1}y, \qquad W \cap f^{-1}f^{\#}U, \qquad W \cap f^{-1}f^{\#}V,$$

each of which is contained in U^y. Indeed, we have

$$W \cap f^{-1}y \subset U \cap f^{-1}y \subset U^y,$$

$$W \cap f^{-1}f^{\#}U \subset f^{-1}f^{\#}U \subset U^y,$$

$$W \cap f^{-1}f^{\#}V \subset W \cap V = \emptyset \subset U^y.$$

Thus, the set $W \cap f^{-1}H$ is a neighborhood of x, contained in U^y. Lemma 2 is proved.

It follows from Lemma 2 that the sets $\widetilde{A}_s^{\gamma}$ are open. The system $\tilde{\alpha}_\gamma$ is a cover of the set $f^{-1}B_\gamma$. Indeed:

$$\bigcup_{s=1}^{r(\gamma)} \widetilde{A}_s^{\gamma} = f^{-1}B_\gamma \cap \left(\bigcup_{s=1}^{r(\gamma)} \{(A_s^{y_\gamma} \cap f^{-1}y_\gamma) \cup f^{-1}f^{\#}A_s^{y_\gamma}\} \right)$$

$$= f^{-1}B_\gamma \cap \left(f^{-1}y_\gamma \cup \left(\bigcup_{s=1}^{r(\gamma)} f^{-1}f^{\#}A_s^{y_\gamma} \right) \right) = f^{-1}B_\gamma \cap f^{-1} \left(\{y_\gamma\} \cup \left(\bigcup_{s=1}^{r(\gamma)} f^{\#}A_s^{y_\gamma} \right) \right)$$

$$= f^{-1}B_\gamma \cap f^{-1}U_{y_\gamma} = f^{-1}B_\gamma.$$

Since the system $\tilde{\alpha}_\gamma$ is obtained from the system α_{y_γ} by reducing its elements, $\tilde{\alpha}_\gamma$ is inscribed in α_{y_γ}, and the multiplicity of $\tilde{\alpha}_\gamma$ is no greater than $l+1$. The system

$$\tilde{\alpha} = \bigcup_{\gamma \in \Gamma} \tilde{\alpha}_\gamma$$

is a cover of the space X, inscribed in the cover ω.

Denote by Γ_1 the set of those indices $\gamma \in \Gamma$ for which $y_\gamma \in B_\gamma$. Put also

$$Y_1 = \{y_\gamma \mid \gamma \in \Gamma_1\}.$$

Then, for every point $y \in Y \setminus Y_1$, the set $f^{-1}y$ is contained in some element of the cover $\tilde{\alpha}$, and, consequently, in some element of the cover ω. This means that the system

$$f^{\#}\omega = \{f^{\#}O_p,\ p = 1, \dots, q\}$$

is a cover of the set $Y \setminus Y_1$. Since to each point $y_\gamma \in Y_1$ there corresponds an element B_γ of the locally finite cover β containing it, the set Y_1 is a discrete subspace of the space Y. For each $\gamma \in \Gamma_1$ there exist open sets W_γ, V_γ such that

$$y_\gamma \in W_\gamma \subset [W_\gamma] \subset V_\gamma \subset B_\gamma$$

and the system $\{V_\gamma \mid \gamma \in \Gamma_1\}$ is disjoint.

For $\gamma \in \Gamma_1$, by α'_γ we denote the restriction of the system $\tilde{\alpha}_\gamma$ to the set $f^{-1}V_\gamma$. The bodies of different systems α'_γ do not intersect. Hence, the system

$\alpha' = \bigcup_{\gamma\in\Gamma_1} \alpha'_\gamma$ is a covering (generally speaking, infinite) of multiplicity $\leq l+1$ of the set $f^{-1}V \left(V = \bigcup_{\gamma\in\Gamma_1} V_\gamma\right)$, inscribed in the covering ω. Refining the system α' with respect to the covering ω, we obtain a covering $\omega^* = \{O^*_p, p = 1, \dots, q\}$ of multiplicity $\leq l+1$. Here the refinement is understood in the following sense:

$$O^*_p = \bigcup \{A \in \alpha' \mid A \subset O_p,\ A \not\subset O_k \text{ for } k < p\}.$$

Since for each set $A \in \alpha'$ we have $f^{\#}A \supset fA \setminus Y_1$, it follows that $f^{\#}O^*_p \supset fO^*_p \setminus Y_1$ for $p = 1, \dots, q$. Hence the system $f^{\#}\omega^* = \{f^{\#}O^*_p \mid p = 1, \dots, q\}$ is a covering of multiplicity $\leq l+1$ of the set $V \setminus Y_1$. The set $Y \setminus W$ $\left(W = \bigcup_{\gamma\in\Gamma_1} W_\gamma\right)$ is closed in Y and does not intersect Y_1.

Consequently, the system $f^{\#}\omega$ is a covering of the set $Y \setminus W$. By Lemma 1, there exists a finite system $\tau = \{T_r \mid r \in R\}$ of sets open in Y, of multiplicity $\leq l+1$, covering the set $Y \setminus W$ and inscribed in the system $f^{\#}\omega$. Moreover, one may assume that every element of the system τ that intersects $[W]$ is contained in some element of the system $f^{\#}\omega^*$ (it suffices to regard the system τ as inscribed in the covering $\{f^{\#}\omega \cap (Y \setminus [W])\} \cup f^{\#}\omega^*$ of the set $Y \setminus W)^*$.

Since the system τ is inscribed in the system $f^{\#}\omega$, the system $f^{-1}\tau = \{f^{-1}T_r \mid r \in R\}$ is inscribed in the covering ω, and, like τ, has multiplicity $\leq l+1$. The system $f^{-1}\tau$ covers the set $f^{-1}(Y \setminus W) = X \setminus f^{-1}W$. The system ω^* covers the set $f^{-1}V \supset f^{-1}W$. Consequently, the system $\omega^* \cup f^{-1}\tau$ is a covering of the space X. The systems ω^* and $f^{-1}\tau$ are inscribed in ω, hence the covering $\omega^* \cup f^{-1}\tau$ is inscribed in ω. Put now $\tilde{\omega}^* = \omega^* \cap f^{-1}W$. Since the system $f^{-1}\tau$ covers the set $X \setminus f^{-1}W$, the system $\tilde{\omega}^* \cup f^{-1}\tau$ will be a covering of the space X, also inscribed in the covering ω. Now refine the covering $\tilde{\omega}^* \cup f^{-1}\tau$ in order to obtain a covering of multiplicity $\leq l+1$. We have $\omega^* = \{O^*_1, \dots, O^*_q\}$, where $\widetilde{O}^*_p = O^*_p \cap f^{-1}W$, $p = 1, \dots, q$.

Put

$$\widehat{O}_p = \widetilde{O}_p^* \cup \left(\bigcup\{f^{-1}T_r \in f^{-1}\tau \mid f^{-1}T_r \subset O_p^*,\ f^{-1}T_r \not\subset O_k^* \text{ for all } k < p\}\right).$$

Each set $\widehat{O}_p$ is contained in the set O_p^*, and therefore the system $\hat{\omega} = \{\widehat{O}_p \mid p = 1, \dots, q\}$ is inscribed in the covering ω. Put

$$S = \{s \in R \mid \text{there does not exist a number } p,\ 1 \le p \le q, \text{ such that } f^{-1}T_s \subset \widehat{O}_p\},$$

and let $\tau' = \{T_s \in \tau \mid s \in S\}$. Then the system $\hat{\omega} \cup f^{-1}\tau'$ is a covering of the space X.

Indeed, the system $\hat{\omega} \cup f^{-1}\tau'$ lacks only those elements T_r of the covering $\tilde{\omega}^* \cup f^{-1}\tau$ which are contained in some elements $\widehat{O}_p$ of the system $\hat{\omega}$. We shall show that the multiplicity of the covering $\hat{\omega} \cup f^{-1}\tau'$ does not exceed $l+1$. For this, note first that the body of the system $f^{-1}\tau'$ does not intersect the set $f^{-1}[W]$. Indeed, if for some $r \in R$ the set $f^{-1}T_r$ intersects $f^{-1}[W]$, i.e. $T_r \cap [W] \neq \emptyset$, then, by one of the properties of the system τ, the set T_r is contained in some element $f^{\#}O_p^*$ of the system $f^{\#}\omega^*$. Then the set $f^{-1}T_r$ is contained in O_p^*. Let p_0 be the least of those numbers p for which $f^{-1}T_r \subset O_p^*$. Then $f^{-1}T_r \subset \widehat{O}_{p_0}$, and therefore, by the definition of the set S, we have $r \in R \setminus S$. Thus, none of the elements of the system $f^{-1}\tau'$ intersects the set $f^{-1}[W] \supset f^{-1}W$. Let $x \in f^{-1}W$. Then the point x can be contained only in those elements of the covering $\hat{\omega} \cup f^{-1}\tau'$ which have the form $\widehat{O}_p$. But the restriction of the system $\hat{\omega}$ to the set $f^{-1}W$ coincides with the system $\tilde{\omega}^*$. Consequently, the point x is contained in no mo-

* If α is a system of subsets of a set E and $F \subset E$, then by $\alpha \cap F$ is denoted the restriction of the system α to F.

fewer than $l+1$ elements of the cover $\hat{\omega} \cup f^{-1}\tau'$. On the other hand, the restriction of the system $\hat{\omega} \cup f^{-1}\tau'$ to the set $X \setminus f^{-1}W$ is a coarsening of the system $f^{-1}\tau \cap (X \setminus f^{-1}W)$, which has multiplicity $\le l+1$. Hence, also on the set $X \setminus f^{-1}W$, the cover $\hat{\omega} \cup f^{-1}\tau'$ has multiplicity $\le l+1$. As for the fineness of the cover $\hat{\omega} \cup f^{-1}\tau'$, both the system $\hat{\omega}$ and the system $f^{-1}\tau'$ are inscribed in the cover ω. Thus, $\hat{\omega} \cup f^{-1}\tau'$ is a finite cover of the space X which has multiplicity $\le l+1$ and is inscribed in the cover ω. The theorem is proved.

Department of Mechanics and Mathematics
of Moscow State University
named after M. V. Lomonosov

Received
26 VI 1968

CITED LITERATURE

1. E. G. Sklyarenko, Bull. Acad. Polon. Sci., Ser. Math., **10**, No. 8, 429 (1962).

2. B. A. Pasynkov, Vestn. Moskovsk. Univ., Mathematics, Mechanics, No. 4, 3 (1965).

Note: Figure translations are in progress. See original paper for figures.