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MATHEMATICS

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ON A DECOMPOSITION OF A LINEAR OPERATOR

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Let X be a vector space over a field K; $H(X)$ the set of all linear operators acting in X. Denote by $D(A)$ and $R(A)$, respectively, the domain of definition and the range of the operator $A \in H(X)$. Put

$$Z(A) = \{x : \ x \in D(A), \ Ax = \theta\},$$

where θ is the zero element of the space X;

$$N(A) = \bigcup_{n=1}^{\infty} Z(A^n);$$

$A_\lambda = \lambda E - A$, where E is the identity operator on X, $\lambda \in K$;

$$\sigma(A) = \{\lambda : \ \lambda \in K, \ Z(A_\lambda) \neq \{\theta\}\};$$

$$\mathcal{E}(K_0; A) = \bigoplus_{\lambda \in K_0} N(A_\lambda),$$

where K_0 is an arbitrary nonempty part of K. An element x of $\bigcap_{n=1}^{\infty} D(A^n)$ is called A-algebraic if the linear subspace $\mathcal{L}(x; A)$, generated by the elements $x, Ax, \dots, A^n x, \dots$, has finite dimension. The operator A is called locally algebraic if the set $M(A)$ of all A-algebraic elements is equal to $D(A)$.

Let $A \in H(X)$, $K_0 \neq K$.

Theorem 1 (on the decomposition of an operator). For every μ from $K \setminus K_0$ there exists a decomposition $A = B + C$, where $B, C \in H(X)$, and they have the following properties: a) $\sigma(B) \cap K_0 = \emptyset$; b) $D(C) = X$; c) $R(C) = \mathcal{E}(K_0; A)$; d) C is a locally algebraic operator; e) $BC = \mu C$.

Theorem 2 (on an invariant complement—necessary conditions). If in X there exists an A-invariant algebraic complement to $\mathcal{E}(K_0; A)$, then for any μ from

$K \setminus K_0$ a decomposition $A = B + C$ is possible, where $B, C \in H(X)$, and they have the properties b), c), d), e) (see Theorem 1), and also a$'$) $\sigma(A) \setminus K_0 \subseteq \sigma(B) \subseteq \{\mu\} \cup \sigma(A) \setminus K_0$ and f) $CB = \mu C$.

Theorem 3 (on an invariant complement–sufficient conditions). If, for some μ from $K \setminus K_0$, the operator A admits a decomposition $A = B + C$, where $B, C \in H(X)$ and have the properties a), b), d), e) and f) (see Theorems 1 and 2), then in X there exists an A-invariant algebraic complement to $\mathcal{E}(K_0; A)$.

Theorem 4 (on normal separability). Suppose that for some μ from $K \setminus K_0$ there is a decomposition $A = B + C$, where $B, C \in H(X)$ and they have properties a), b), d), e) and f) (the condition of Theorem 3). If F is some A-invariant algebraic complement in X to $\mathcal{E}(K_0; A)$ and $\lambda \in K_0$, then the equality

$$A_\lambda(D(A) \cap F) = F$$

holds if and only if

$$R(B_\lambda) = X.$$

For the proof of these theorems we first give several auxiliary propositions. Let $A, B, C \in H(X)$, $\lambda, \mu, \nu \in K$.

I. If $BC = \mu C$, then $R(C) \subseteq D(B)$ and $Bx = \mu x$ for $x \in R(C)$.

II. Let $A = B + C$, $T = \mu E + C$. If $D(A) = D(B)$ and $BC = \mu C$, then $Ax = Tx$ for $x \in R(C)$.

III. If $A = B + C$, $T = \mu E + C$, $D(C) = X$ and $BC = \nu C = CB$, then $D(AT) = D(A)$, $D(TA) = D(A)$ and $ATx = TAx$ for $x \in D(A)$.

IV. If $A = B + C$, $Z(B) = \{\theta\}$ and $BC = \nu C$ $(\nu \neq 0)$, then $N(A) \subseteq R(C)$.

Proof. It is required to show that $Z(A^n) \subseteq R(C)$ $(n = 1, 2, ...)$. We shall establish this by induction. Let $x \in Z(A)$, i.e. $Bx + Cx = \theta$. Since $Cx = \nu^{-1}BCx$, we have $B(x + \nu^{-1}Cx) = \theta$; hence, taking into account the condition $Z(B) = \{\theta\}$, we obtain $x + \nu^{-1}Cx = \theta$, $x = -\nu^{-1}Cx \in R(C)$. This proves the inclusion $Z(A) \subseteq R(C)$. Suppose now that $Z(A^n) \subseteq R(C)$ and $x \in Z(A^{n+1})$; then $Ax \in Z(A^n) \subseteq R(C)$, i.e. $Bx + Cx \in R(C)$, whence $Bx \in R(C)$, i.e. $Bx = Cx_1$. Further we have: $Bx = \nu^{-1}BCx_1$, $B(x - \nu^{-1}Cx_1) = \theta$, $x = \nu^{-1}Cx_1 \in R(C)$, which proves the inclusion $Z(A^{n+1}) \subseteq R(C)$.

V. If $A = B + C$, $T = \nu E + C$ $(\nu \neq 0)$, $D(A) = D(B)$, $Z(B) = \{\theta\}$, and $BC = \nu C$, then $N(A) = N(T)$.

Proof. Consider the restrictions A_0 and T_0 of the operators A and T to $R(C)$ (according to I, $R(C) \subseteq D(A)$). Since $N(A) \subseteq R(C)$ and $N(T) \subseteq R(C)$ (see IV), we have $N(A_0) = N(A)$ and $N(T_0) = N(T)$. It remains to use the equality $A_0 = T_0$ (see II).

VI. If C is a locally algebraic operator and $Z(C) = \{\theta\}$, then $R(C) = D(C)$.

VII. If C is a locally algebraic operator and $D(C) = X$, then for any $K_0 \subseteq K$ in X there exists a C-invariant algebraic complement to $\mathcal{E}(K_0; C)$.

The proof can be obtained by somewhat modifying the proof of Theorem 5 in ([1]).

VIII. If $A_\lambda x \in \mathcal{E}(K_0; A)$, where $\lambda \in K_0$, then $x \in \mathcal{E}(K_0; A)$.

Proof. The condition $A_\lambda x \in \mathcal{E}(K_0; A)$ means that

$$A_\lambda x = \sum_{k=1}^{n} x_k, \quad \text{where } x_k \in N(A_{\lambda_k}),\ \lambda_k \in K_0\ (k = 1, \dots, n).$$

Let $A_{\lambda_k}^{r_k} x_k = \theta$ $(k = 1, \dots, n)$; then

$$\left(\prod_{k=1}^{n} A_{\lambda_k}^{r_k} \right) A_\lambda x = \theta,$$

which means $x \in M(A)$. Denote by $\mathcal{E}_1$ an A-invariant algebraic complement in $M(A)$ to $\mathcal{E}(K_0; A)$ (see VII), and let $x = u + v$, where $u \in \mathcal{E}(K_0; A)$, $v \in \mathcal{E}_1$; then $A_\lambda v = A_\lambda x - A_\lambda u \in \mathcal{E}(K_0; A)$. But $A_\lambda v \in \mathcal{E}_1$; consequently, $A_\lambda v = \theta$, and, since $\lambda \in K_0$, $v \in \mathcal{E}(K_0; A)$. At the same time $v \in \mathcal{E}_1$, hence $v = \theta$, $x = u \in \mathcal{E}(K_0; A)$.

IX. Let $A = B + C$, $R(C) \subseteq D(B)$, $B(R(C)) \subseteq R(C)$. If the restriction B_0 of the operator B to $R(C)$ is a locally algebraic operator and $Z(B_0) = \{\theta\}$, then $R(A) \subseteq R(B)$.

Proof. From the equality $A = B+C$ follows the inclusion $R(A) \subseteq R(B)+R(C)$. But $R(B) + R(C) = R(B)$, since $R(B) \supseteq R(C)$ $(R(B) \supseteq R(B_0)$, and according to VI $R(B_0) = D(B_0)(= R(C)))$.

X. Let $B_1, B_2, C_1, C_2 \in H(X)$, $Z(B_1) = Z(B_2) = \{\theta\}$, $D(C_1) = D(C_2) = X$, $B_1 + C_1 = B_2 + C_2$, $B_1 C_1 = \nu C_1$, $B_2 C_2 = \nu C_2$. If C_1 and C_2 are locally algebraic operators, then $R(B_1) = R(B_2)$.

Proof. We shall show that $R(B_2) \subseteq R(B_1)$. For this, consider the equality $B_2 = B_1 + C$, where $C = C_1 - C_2$, and apply proposition IX, putting $A = B_2$, $B = B_1$.

It is necessary to verify the three conditions appearing in IX: 1) $R(C) \subseteq D(B_1)$; 2) $B_1(R(C)) \subseteq R(C)$; 3) the restriction of the operator B_1 to $R(C)$ is a locally algebraic operator. Condition 1) follows from the inclusions $R(C_1) \subseteq D(B_1)$, $R(C_2) \subseteq D(B_2)$ (see I) and the equality $D(B_1) = D(B_2)$, which holds by virtue of the hypotheses $B_1+C_1 = B_2+C_2$, $D(C_1) = D(C_2) = X$. Condition 2) follows from the easily proved equality $B_1 C = C(\nu E+C_2)$. To verify condition 3), write an arbitrary element of $R(C)$ in the form $y = Cx$ and consider $\mathcal{L}(y; B_1)$. It is not hard to see, using the equality $B_1 C = C(\nu E + C_2)$, that $B_1^n y = C(\nu E + C_2)^n x$

$(n = 1, 2, ...)$. Hence we conclude that $\mathcal{L}(y; B_1) = C(\mathcal{L}(x; \nu E + C_2))$. Since the operator $\nu E + C_2$ is locally algebraic (as the sum of commuting locally algebraic-

···operators νE and C_2, then $\dim \mathcal{L}(x; \nu E + C_2) < \infty$, and hence $\dim \mathcal{L}(y; B_1) < \infty$. Thus condition 3) is verified. Consequently, $R(B_2) \subseteq R(B_1)$. The converse inclusion, of course, also holds.

Proof of Theorem 1. To construct the desired decomposition of the operator A, take in X some algebraic complement F to the subspace $\mathcal{E}(K_0; A)$. Let P and Q be the projectors generated by the decomposition

$$X = \mathcal{E}(K_0; A) \oplus F,$$

where $R(P) = \mathcal{E}(K_0; A)$, $R(Q) = F$. Put $B = AQ + \mu P$, $C = AP - \mu P$, where μ is a prescribed value from $K \setminus K_0$, and show that the operators B and C have the required properties. Properties b) and e) are verified easily. We prove a), c), and d).

a) It is necessary to show that $Z(B_\lambda) = \{\theta\}$ if $\lambda \in K_0$. Let $B_\lambda x = \theta$, i.e.,

$$AQx + \mu Px = \lambda x;$$

then $A_\lambda Qx = (\mu - \lambda)Px \in \mathcal{E}(K_0; A)$. By VIII, $Qx \in \mathcal{E}(K_0; A)$. But $Qx \in F$; hence $Qx = \theta$. Therefore, in view of the equality $A_\lambda Qx = (\mu - \lambda)Px$ and the condition $\mu - \lambda \neq 0$, we obtain $Px = \theta$. Hence $Px + Qx = x = \theta$.

b) Let $\lambda \in K_0$; then $\lambda - \mu = \nu \neq 0$ and, by what has just been proved, $Z(B_\lambda) = \{\theta\}$. Since $A_\lambda = B_\lambda - C$ and $B_\lambda(-C) = \nu(-C)$, by IV, $N(A_\lambda) \subseteq R(C)$. Consequently, $\mathcal{E}(K_0; A) \subseteq R(C)$. The reverse inclusion is obvious.

c) Let $y = Cx$; then $y \in \mathcal{E}(K_0; A)$, $Py = y$, $Cy = (A - \mu E)y$,

$$C^n y = \sum_{k=0}^{n} (-\mu)^k \binom{n}{k} A^{n-k} y \quad (n = 1, 2, ...).$$

Hence we conclude that

$$\mathcal{L}(Cy; C) \subseteq \mathcal{L}(y; A).$$

But $\dim \mathcal{L}(y; A) < \infty$; therefore, $\dim \mathcal{L}(x; C) < \infty$ for every $x \in X$.

Proof of Theorem 2. The required decomposition of the operator A is effected by the operators B and C, which are constructed in the same way as in Theorem 1, but in the present case the algebraic complement F to $\mathcal{E}(K_0; A)$ is chosen to be A-invariant, owing to which the additional properties a′) and f) appear (note that a′) ⇒ a)). Property f) is verified easily. We prove a′). Let $\lambda \in \sigma(A) \setminus K_0$, $A_\lambda x = \theta$, $x \neq \theta$; then $x \notin \mathcal{E}(K_0; A)$, $Qx = \nu \neq \theta$, $B_\lambda \nu = A_\lambda \nu (= A_\lambda Qx)$. Since $AQ = QA$ (by the A-invariance of $\mathcal{E}(K_0; A)$ and F), we have $A_\lambda Qx = QA_\lambda x = \theta$, hence $B_\lambda \nu = \theta$, i.e. $\lambda \in \sigma(B)$. Let now $\lambda \in \sigma(B)$, $B_\lambda x = \theta$, $x \neq \theta$, $\nu = Qx$; then

$$A_\lambda \nu = B_\lambda x + (\mu - \lambda)Px = (\mu - \lambda)Px.$$

If $Px = \theta$, then $A_\lambda \nu = \theta$ and $\nu = x \neq \theta$; consequently, $\lambda \in \sigma(A)$. If $Px \neq \theta$, then, applying the operator P to the equality

$$A_\lambda Qx = (\mu - \lambda)Px,$$

we obtain $(\mu - \lambda)Px = \theta$, whence $\lambda = \mu$. Thus,

$$\sigma(B) \subseteq \{\mu\} \cup \sigma(A).$$

But $\sigma(B) \cap K_0 = \emptyset$; hence

$$\sigma(B) \subseteq \{\mu\} \cup \sigma(A) \setminus K_0.$$

Proof of Theorem 3. Let $T = \mu E + C$; then $T_\lambda = (\lambda - \mu)E - C$, and since $A_\lambda = B_\lambda - C$, $B_\lambda(-C) = (\lambda - \mu)(-C)$, $Z(B_\lambda) = \{\theta\}$ for $\lambda \in K_0$, it follows, by V, that $N(A_\lambda) = N(T_\lambda)$ for every $\lambda \in K_0$. Consequently, $\mathcal{E}(K_0; A) = \mathcal{E}(K_0; T)$. Let F be some T-invariant algebraic complement in X to $\mathcal{E}(K_0; T)$ (see VII). We shall establish the inclusion $A(D(A) \cap F) \subseteq F$, whereby the theorem will be proved. Fix $x_0 \in D(A) \cap F$. Since

$$T(D(A)) \subseteq \mu E(D(A)) + C(D(A)) \subseteq D(A) + R(C),$$

and $R(C) \subseteq D(B)(= D(A))$ (see I), we have $T(D(A)) \subseteq D(A)$; hence, taking into account the T-invariance of F, we obtain the inclusion

$$\mathcal{L}(x_0; T) \subseteq D(A) \cap F.$$

Choose a basis $x_1, \dots, x_n$ in $\mathcal{L}(x_0; T)$ and put

$$Ax_k = y_k = y'_k + y''_k,$$

where $y'_k \in \mathcal{E}(K_0; T)$, $y''_k \in F$ $(k = 1, \dots, n)$. Since $y'_k \in \mathcal{E}(K_0; A)$, there exists an operator S of the form

$$\prod_{i=1}^{m} T_{\lambda_i}^{m_i},$$

where $\lambda_i \in K_0$ $(i = 1, \dots, m)$, such that $Sy'_k = \theta$. Therefore, $Sy_k = Sy''_k \in F$ $(k = 1, \dots, n)$. It is easy to see, using Proposition VIII, that $Z(S) \subseteq \mathcal{E}(K_0; T)$, and hence

$$Z(S) \cap \mathcal{L}(x_0; T) = \{\theta\}$$

(since $\mathcal{L}(x_0; T) \subseteq D(A) \cap F$). From this we conclude that

$$S(\mathcal{L}(x_0; T)) = \mathcal{L}(x_0; T),$$

from which it follows that the equation

$Sx = x_0$ has a solution $x \in \mathcal{L}(x_0; T)$. Writing x in the form $\sum_{k=1}^{n} \alpha_k x_k$, we find

$$SAx = \sum_{k=1}^{n} \alpha_k SAx_k = \sum_{k=1}^{n} \alpha_k Sy_k \in F.$$

But $SAx = ASx$ for $x \in D(A)$, since $TAx = ATx$ for $x \in D(A)$ (see III); consequently, $Ax_0 = ASx = SAx \in F$. In view of the arbitrariness of $x_0 \in D(A) \cap F$, this proves that $A(D(A) \cap F) \subseteq F$.

Proof of Theorem 4. Suppose first that the given decomposition $A = B + C$ has been obtained in the way it was done in the proof of Theorem 2: $B = AQ + \mu P,\ C = AP - \mu P$.

Let the equality $A_\lambda(D(A) \cap F) = F$ be given. It is required to prove that the equation $B_\lambda x = y$ has a solution for every $y \in X$. To this end take $\bar{x} \in D(A) \cap F$ so that $A_\lambda \bar{x} = Qy$. Then, taking into account that $QP = 0,\ P^2 = P,\ P\bar{x} = \theta$, and $Q\bar{x} = \bar{x}$, we obtain:

$$B_\lambda(Py/(\mu - \lambda) + \bar{x}) = [A_\lambda Q + (\mu - \lambda)P](Py/(\mu - \lambda) + \bar{x}) = A_\lambda \bar{x} + Py = Qy + Py = y.$$

Consequently, the required solution is $Py/(\mu - \lambda) + \bar{x}$.

Let the equality $R(B_\lambda) = X$ be given. It is required to prove that the equation $A_\lambda x = y$ has a solution for every $y \in F$. Take $\bar{x} \in D(B_\lambda)(= D(A))$ so that $B_\lambda \bar{x} = y$. Then we obtain $Q\bar{x} \in D(A) \cap F$,

$$A_\lambda Q\bar{x} = B_\lambda \bar{x} - (\mu - \lambda)P\bar{x} = y - (\mu - \lambda)P\bar{x}.$$

Since $A_\lambda Q\bar{x} \in F$ and $y \in F$, it follows that $(\mu - \lambda)P\bar{x} \in F$. Consequently, $P\bar{x} = \theta,\ A_\lambda Q\bar{x} = y$. Thus the required solution is $Q\bar{x}$.

It remains to remove the assumption made at the beginning of the proof. For this it is necessary to take into account VIII and the fact that the operator A_λ does not depend on the choice of the decomposition $A = B + C$.

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Note: Figure translations are in progress. See original paper for figures.