ON A VECTOR SPACE CONNECTED WITH HADWIGER’ S CONJECTURE

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Abstract

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ON A VECTOR SPACE CONNECTED WITH HADWIGER' S CONJECTURE

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We consider here only ordinary (i.e., finite, undirected, without loops or multiple edges) graphs; by $n(L) = |X|$ and $m(L) = |U|$ we denote, respectively, the numbers of vertices and edges of the graph $L = (X, U)$, and by F_n the complete graph with n vertices.

A graph L is called **piecewise complete** if all its connected components are complete graphs. To all possible piecewise complete graphs, considered up to isomorphism, there correspond one-to-one all possible partitions (in the sense of ([3])), each of which is symbolically written in the form $\lambda_1^{\alpha_1}, \lambda_2^{\alpha_2} \dots \lambda_s^{\alpha_s}$, where α_i is the number of those components that have λ_i vertices.

Let F be the associative-commutative ring of polynomials in a countable set of variables $z_1, z_2, \dots$ over the field R of rational numbers. To each graph L we assign an element $f(L) \in F$, where the function f is defined recursively as follows:

a) if L is piecewise complete, corresponding to the partition $\lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \dots \lambda_s^{\alpha_s}$, then

$$f(L) = z_{\lambda_1}^{\alpha_1} z_{\lambda_2}^{\alpha_2} \dots z_{\lambda_s}^{\alpha_s};$$

b) if L is not piecewise complete, then

$$f(L) = \frac{1}{m(L)} \sum_{u \in U} [f(L_u) - f(L^u)], \tag{1}$$

where L_u and L^u denote the graphs obtained from L, respectively, by deleting and contracting the edge u (under contraction, the edge itself is deleted, its end vertices are identified, and each bundle of multiple edges, if it arises, is replaced by a single edge).

From the definition of the function $f(L)$ it is clear that if the graph L' is isomorphic to L, then $f(L') = f(L)$. Evidently, $f(F_n) = z_n$.

The quantity

$$\eta\left(z_{\lambda_1}^{\alpha_1} z_{\lambda_2}^{\alpha_2} \dots z_{\lambda_s}^{\alpha_s}\right) = \max_{1 \le i \le s} \lambda_i$$

will be called the **Hadwiger number** of the monomial $z_{\lambda_1}^{\alpha_1} z_{\lambda_2}^{\alpha_2} \dots z_{\lambda_s}^{\alpha_s}$. The Hadwiger number $\eta(f)$ of an arbitrary element $f \in F$ is defined as the largest of the Hadwiger numbers of all monomials appearing in the representation of f as a reduced polynomial; since such a representation is unique, the function $\eta(f)$ maps the ring F into the field R uniquely.

As is known, for a graph L its Hadwiger number $\eta(L)$ is, by definition, the number of vertices of the largest complete graph into which L can be transformed by means of contracting edges, deleting edges, and deleting vertices.

Theorem 1. $\eta(L) = \eta(f(L))$.

Indeed, since in (1) the summation is over all edges of the graph, after the complete use of rules a) and b), but without bringing like terms together, we obtain for $f(L)$ an expression in the form of a sum of monomials corresponding to all those piecewise complete graphs into which L can be transformed by contracting and deleting edges, without the appearance of piecewise complete graphs at intermediate stages. By virtue of the definition of $\eta(L)$, among these monomials there is not one with Hadwiger number $> \eta(L)$, but at least one has Hadwiger number $\eta(L)$. Monomials of the latter type cannot mutually cancel in the sum $f(L)$, for from the fact that in relation (1) the graph L_u has the same number of vertices as L, while the graph L^u has one fewer vertex, it follows that the signs of the coefficients of such monomials are identical.

Let a monomial

$$z_{\lambda_1}^{\alpha_1} z_{\lambda_2}^{\alpha_2} \cdots z_{\lambda_s}^{\alpha_s}$$

be written without using powers, i.e. in the form $z_{l_1} z_{l_2} \cdots z_{l_p}$, where identical factors are allowed. To the monomial assign the system of numbers

$$r_i(l_1, l_2, \dots, l_p) = r_i(z_{l_1} z_{l_2} \cdots z_{l_p})$$

$(i = 1, 2, \dots)$, defined by means of the identities

$$i! \sum_{l_1, l_2, \dots, l_p} \frac{r_i(l_1, l_2, \dots, l_p)}{l_1!\, l_2! \cdots l_p!} z_1^{l_1} z_2^{l_2} \cdots z_p^{l_p} = \left[\prod_{j=1}^{p} (1 + z_j) - 1\right]^i; \tag{2}$$

in other words, $r_i(l_1, l_2, \dots, l_p)$ is the product of the number $l_1! l_2! \cdots l_p!/i!$ by the coefficient of $z_1^{l_1} z_2^{l_2} \cdots z_p^{l_p}$ in the polynomial on the right-hand side. For an arbitrary element f ∈ F we define the numbers r_i(f) as follows: let

$$\mathrm{f} = \sum_k a_k \mathrm{f}_k$$

be the representation of f in the form of a reduced polynomial; then

$$r_i(\mathrm{f}) = \sum_k a_k r_i(\mathrm{f}_k).$$

Following (2), we denote by $r_i(L)$ the number of different proper colorings of the vertices of the graph L by means of i colors ($i = 1, 2, \dots$).

Theorem 2. $r_i(L) = r_i(f(L))$.

Proof. We first consider the case when L is a piecewise complete graph with p components $F_{l_1}, F_{l_2}, \dots, F_{l_p}$ ($p \geqslant 1$; some of the numbers $l_1, l_2, \dots, l_p$ may coincide with one another). For $p = 1$ the assertion of the theorem for such graphs is verified directly: if $L = F_{l_1}$, then $r_i(L) = \delta^i_{l_1}$ (1 for $i = l_1$ and 0 for $i \neq l_1$), and the number $r_i(l_1)$, as follows from (2), is also equal to $\delta^i_{l_1}$. Suppose that for some $p \geqslant 1$ it has already been proved that the equality of the theorem is valid for any piecewise complete graph with p components (and for any $i \geqslant 1$), and let L be an arbitrary piecewise complete graph having $p + 1$ components $F_{l_1}, F_{l_2}, \dots, F_{l_p}, F_{l_{p+1}}$; we shall show that $r_i(L)$ coincides with the number $r_i(l_1, l_2, \dots, l_p, l_{p+1})$, defined from the identity (2), i.e. that the identity

$$i! \sum_{l_1, l_2, \dots, l_p, l_{p+1}} \frac{r_i(L)}{l_1!\, l_2! \cdots l_p!\, l_{p+1}!} z_1^{l_1} z_2^{l_2} \cdots z_p^{l_p} z_{p+1}^{l_{p+1}} = \left[\prod_{j=1}^{p+1} (1 + z_j) - 1\right]^i. \tag{3}$$

holds. Denote by L' the piecewise complete graph formed by the p components $F_{l_1}, F_{l_2}, \dots, F_{l_p}$ of the graph L. By Theorem 7 of the paper ([1]) we have

$$r_i(L) = \sum_q \sum_t r_q(L')\, r_t(F_{l_{p+1}}) \frac{q!\, t!}{(i-q)!(i-t)!(q+t-i)!} =$$

$$= \sum_q r_q(L') \frac{q!\, l_{p+1}!}{(i-q)!(i-l_{p+1})!(q+l_{p+1}-i)!};$$

substituting this expression into the left-hand side of the identity (3) being proved, we transform it as follows:

$$i! \sum_{l_1, l_2, \dots, l_p, l_{p+1}} \frac{r_i(L)}{l_1! l_2! \dots l_p! l_{p+1}!} z_1^{l_1} z_2^{l_2} \cdots z_p^{l_p} z_{p+1}^{l_{p+1}} =$$

$$= i! \sum_{l_{p+1}} \frac{z_{p+1}^{l_{p+1}}}{l_{p+1}!} \sum_{l_1, l_2, \ldots, l_p} \frac{z_1^{l_1} z_2^{l_2} \cdots z_p^{l_p}}{l_1! l_2! \ldots l_p!} \sum_q \frac{r_q(L') q! \, l_{p+1}!}{(i-q)!(i-l_{p+1})!(q+l_{p+1}-i)!} =$$

$$= \sum_{l_{p+1}} \frac{i! \, z_{p+1}^{l_{p+1}}}{l_{p+1}!(i-l_{p+1})!} \sum_q \frac{l_{p+1}!}{(i-q)!(q+l_{p+1}-i)!} \, q! \sum_{l_1, l_2, \ldots, l_p} \frac{r_q(L')}{l_1! l_2! \ldots l_p!} z_1^{l_1} z_2^{l_2} \cdots z_p^{l_p} =$$

$$= \sum_{l_{p+1}} \binom{i}{l_{p+1}} z_{p+1}^{l_{p+1}} \sum_q \binom{l_{p+1}}{q + l_{p+1} - i} q! \sum_{l_1, l_2, \ldots, l_p} \frac{r_q(L')}{l_1! l_2! \ldots l_p!} z_1^{l_1} z_2^{l_2} \cdots z_p^{l_p}.$$

But, by the induction hypothesis,

$$q! \sum_{l_1, l_2, \ldots, l_p} \frac{r_q(L')}{l_1! l_2! \ldots l_p!} z_1^{l_1} z_2^{l_2} \cdots z_p^{l_p} = \left[\prod_{j=1}^{p} (1 + z_j) - 1 \right]^q,$$

and the transformed expression takes the form

$$\sum_{l_{p+1}} \binom{i}{l_{p+1}} z_{p+1}^{l_{p+1}} \sum_q \binom{l_{p+1}}{q + l_{p+1} - i} \left[\prod_{j=1}^{p} (1 + z_j) - 1 \right]^q =$$

$$= \sum_{l_{p+1}} \binom{i}{l_{p+1}} z_{p+1}^{l_{p+1}} (1 + [\,])^{l_{p+1}} \cdot [\,]^{\,i - l_{p+1}} =$$

$$= [\,]^i \sum_{l_{p+1}} \binom{i}{l_{p+1}} \left\{ \frac{z_{p+1}(1 + [\,])}{[\,]} \right\}^{l_{p+1}} = [\,]^i (1 + \{\ \})^i = ([\,] + z_{p+1}(1 + [\,]))^i =$$

$$= ([\,](1 + z_{p+1}) + z_{p+1})^i = \left(\left[\prod_{j=1}^{p} (1 + z_j) - 1 \right] (1 + z_{p+1}) + z_{p+1} \right)^i =$$

$$= \left[\prod_{j=1}^{p+1} (1 + z_j) - 1 \right]^i,$$

i.e. the identity (3) is indeed valid.

We pass to the case of an arbitrary graph $L = (X, U)$. As is known (see (1) or, for example, (2)),

$$r_i(L) = r_i(L_u) - r_i(L^u)$$

for any edge $u \in U$ and any $i \geq 1$. Hence

$$r_i(L) = \frac{1}{m(L)} \sum_{u \in U} \left[r_i(L_u) - r_i(L^u) \right];$$

comparing the last formula with (1) and taking into account that for piecewise complete graphs the assertion of the theorem has already been proved, we easily arrive at the conclusion that $r_i(L) = r_i(f(L))$ for every $i = 1, 2, ...$ for any graph L.

As is known, the chromatic number of a graph L is

$$\gamma(L) = \min\{i \mid r_i(L) \neq 0\}.$$

We shall call the **chromatic number of an element** $f \in F$ the number

$$\gamma(f) = \min\{i \mid r_i(f) \neq 0\}.$$

Then Theorem 2 immediately implies

Theorem 3. $\gamma(L) = \gamma(f(L))$.

Let $\mathcal{L}$ be the set of all ordinary graphs, and $f(L)$ its image under the mapping $f : \mathcal{L} \to F$. Using Theorems 1 and 3, Had-

···Viger' s theorem that $\eta(L) \geq \gamma(L)$ for any graph L, in the following way: if $f \in f(L)$, then

$$\eta(f) \geq \gamma(f). \tag{4}$$

Thus, if the ring F is regarded as a vector space over the field R, generated by all possible expressions (monomials) of the form

$$z_{\lambda_1}^{\alpha_1} z_{\lambda_2}^{\alpha_2} \cdots z_{\lambda_s}^{\alpha_s},$$

then η and γ will be single-valued mappings of this space into the field R, and the study of Hadwiger' s conjecture reduces to two problems:

1) to characterize those elements $f \in F$ for which (4) is valid;

2) to characterize the elements of the set $f(L)$ in F.

The first of these problems is purely algebraic. But even without a complete solution of both problems, such a splitting of Hadwiger' s conjecture makes it possible gradually to approach its solution by studying a number of auxiliary algebraic problems. Namely, for any property $S(f)$ of elements of F one can formulate the proposition: for every $f \in F$, (4) follows from $S(f)$. Taking as S various properties defined without the aid of graphs and implying membership $f \in f(L)$, or, conversely, following from it, we obtain purely algebraic hypotheses, respectively weaker or stronger than Hadwiger' s conjecture.

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Note: Figure translations are in progress. See original paper for figures.