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Abstract

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ON A VARIANT OF THE PLANE STRESS STATE UNDER FINITE ELASTIC DEFORMATIONS

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1. A plate is considered that is elastically symmetric with respect to the mid-plane and has, in the natural state, a thickness d_0 small in comparison with its other dimensions. The external load acting on the edges of the plate in the final state is uniformly distributed over the thickness of the plate and has no components in the direction of the axis perpendicular to the mid-plane of the plate. In view of the small thickness d of the plate in the deformed state, it is assumed that in the symmetric stress tensor

$$\sigma_{33} = \sigma_{13} = \sigma_{23} = 0 \tag{1}$$

and that the remaining components of the stress tensor and the displacements vary very little over the thickness of the plate, i.e., are functions only of the two coordinates y_1, y_2 of the mid-plane of the plate in the current state.

2. The deformation of elements of the mid-plane, as well as the change in their orientation, is completely determined by the 4 derivatives $u_{i,k}$ of the displacements with respect to the coordinates y_k, or by other independent 4 geometrical parameters. As such parameters one chooses the principal stretches λ_1, λ_2, the angle θ between the first principal direction of deformation in the final state and the axis y_1, and the angle ω of rotation of the first two principal axes of deformation about the third axis y_3 in the transition of the body from the initial state to the final one. Between these two groups of parameters the relations are established

$$2(1 + u_{i,k}) = (\lambda_1^{-1} + \lambda_2^{-1}) \cos\omega \pm (\lambda_1^{-1} - \lambda_2^{-1}) \cos(2\theta - \omega), \quad i = k; \quad i, k = 1, 2,$$

$$-2u_{i,k} = \pm(\lambda_1^{-1} + \lambda_2^{-1}) \sin\omega + (\lambda_1^{-1} - \lambda_2^{-1}) \sin(2\theta - \omega), \quad i = k. \tag{2}$$

Hence, by eliminating the displacements and the derivatives $\omega_{,k}$, one obtains the compatibility equation for the deformation parameters

$$[a(a^{-1}\operatorname{sh}\varepsilon\sin 2\theta)_{,1}]_{,1}+[a(a^{-1}\operatorname{sh}\varepsilon\sin 2\theta)_{,1}]_{,2}+[a(a^{-1}\operatorname{ch}\varepsilon)_{,1}]_{,1}$$
$$+[a(a^{-1}\operatorname{sh}\varepsilon\sin 2\theta)_{,2}]_{,1}-[a(a^{-1}\operatorname{sh}\varepsilon\cos 2\theta)_{,2}]_{,2}+[a(a^{-1}\operatorname{ch}\varepsilon)_{,2}]_{,2}$$
$$-2[(\operatorname{ch}\varepsilon-1)\theta_{,1}]_{,2}+2[(\operatorname{ch}\varepsilon-1)\theta_{,2}]_{,1}=0, \tag{3}$$

where $a=\exp v,\ v=\ln\lambda_1\lambda_2,\ \varepsilon=\ln\lambda_1\lambda_2^{-1}$, and v,ε are the plane invariants of the Hencky tensor.

3. For an ideal elastic isotropic body the stress and deformation tensors are coaxial. Therefore the equilibrium equations can be satisfied by introducing the stress function in the form

$$\Sigma=\frac{1}{2}p(U_{,11}+U_{,22}),\qquad T\cos 2\theta=\frac{1}{2}p(U_{,22}-U_{,11}),$$

$$T\sin 2\theta=-pU_{,12},\qquad \Sigma=\frac{1}{2}(\sigma_1+\sigma_2),\qquad T=\frac{1}{2}(\sigma_1-\sigma_2), \tag{4}$$

where σ_1,σ_2 are the principal stresses, and p is a constant stress characteristic of the particular problem.

By eliminating the angle θ from relations (3), (4) and passing to the complex coordinates $\eta=y_1+iy_2,\ \bar\eta=y_1-iy_2$, one obtains a nonlinear dif-

differential equation with respect to the stress function for arbitrary dependences v,ε on Σ and T

$$\operatorname{Re}\left\{\left[ap\left(a^{-1}\frac{\operatorname{sh}\varepsilon}{T}U_{\eta\eta}\right)_{\eta}\right]_{\eta}+U_{\eta\eta}\left[(\operatorname{ch}\varepsilon)_{\eta}U_{\eta\eta\eta}-(\operatorname{ch}\varepsilon)_{\eta}U_{\eta\eta\bar\eta}\right]-\frac{1}{2}\left[a\left(a^{-1}\operatorname{ch}\varepsilon\right)_{\bar\eta}\right]_{\eta}\right\}=0. \tag{5}$$

The dependences established below

$$v=v(\Sigma,T),\qquad \varepsilon=\varepsilon(\Sigma,T) \tag{6}$$

and relations (4) make it possible to assert that equation (5) contains only one unknown function U. Equation (5), under the appropriate simplifications, reduces to the classical biharmonic equation.

4. The relation between the complex displacement function $W=u_1+iu_2$ and the stress function is given by a formula that is a consequence of relations (2),

$$W_{\bar{\eta}} = -\frac{2p}{T}\,\mathrm{th}\,\frac{\varepsilon}{2}\,U_{\eta\eta}(1 - W_{\eta}). \tag{7}$$

5. The expressions for the principal vector and principal moment of the internal forces applied to an arc of an arbitrary contour, in terms of the stress function, as well as the static boundary conditions for the stress function, do not differ in form from the classical ones [5]. The geometric boundary conditions can be obtained by integrating equation (7).

6. The internal energy in an adiabatic process and the free energy in an isothermal process are identified with the elastic potential A, referred to a unit initial volume of the body, for the generalized stresses $s_{ij} = (1+\Delta)\sigma_{ij}$, so that

$$\delta A = \frac{\partial A}{\partial h_{ij}}\delta h_{ij} = s_{ij}\delta h_{ij}, \tag{8}$$

where h_{ij} are the components of the Hencky tensor [1], $1 + \Delta = \exp h_{ii}$.

Assuming the deviators of the tensors s_{ij} (or σ_{ij}) and h_{ij} to be similar, we represent (8) in the form

$$\delta A = s_0\delta h_0 + 3s\delta h;$$

s_0, s are the normal and shear generalized stresses on an octahedral plane; ${}^1/_3 h_0, h$ are the analogous quantities for the Hencky tensor.

Postulating the law of volume change of the element in the form

$$s_0 = s_0(h_0), \tag{9}$$

we arrive at a dependence of the octahedral generalized shear stress only on the analogous invariant of the Hencky tensor

$$s = s(h). \tag{10}$$

The experimental curves (9), (10) can be approximated, for example, by the functions $s_0 = Kh_0(1+k_3h_0^2+...)$, $s = 2Gh(1+g_3h^2+...)$, which, for infinitesimal deformations, reduce to the classical dependences. The relation between stresses and deformations is determined by the formulas

$$s_{ij} = Kh_0(1 + k_3h_0^2 + ...)\delta_{ij} + 2G(1 + g_3h^2 + ...)(h_{ij} - {}^1/_3 h_0\delta_{ij}).$$

These relations generalize Hooke' s law and Kauderer' s law to the case of finite deformations [4].

Putting, in the last formulas written in the principal axes, $\sigma_3 = 0$, one can obtain the dependences (6)

$$v = t_0(a_1 + a_2 t_0 + a_3 t_0^2 + a_4 t^2 + \dots), \qquad \varepsilon = t(b_1 + b_2 t_0 + b_3 t_0^2 + b_4 t^2 + \dots),$$

$$i_0 = \Sigma/G, \quad t = T/G, \quad a_1 = \frac{1}{3} + \frac{4}{9}G/K, \quad a_2 = -\frac{2}{9}G/K + \frac{8}{27}(G/K)^2, \tag{11}$$

$$a_3 = \frac{4}{9}(1 - \frac{2}{3}k_3)(G/K)^3 - \frac{2}{27}(G/K)^2 - \frac{1}{54}g_3, \quad a_4 = -\frac{1}{18}g_3, \quad b_1 = 1,$$

$$b_2 = -\frac{2}{3}G/K, \quad b_3 = -\frac{2}{9}(G/K)^2 - \frac{1}{18}g_3, \quad b_4 = -\frac{1}{6}g_3.$$

7. The structure of equation (5) makes it possible to find its solution by expansion in a series in powers of the small parameter μ of the stress function

$$U = U^{(0)} + U^{(1)}\mu + U^{(2)}\mu^2 \dots, \qquad \mu = p/G.$$

In this case, for each approximation one obtains a biharmonic equation with a right-hand side depending on the preceding approximations

$$U^{(k)}_{\eta\eta\bar\eta\bar\eta} = L^{(k)}(U^{(0)}, \dots, U^{(k-1)}). \tag{12}$$

Thus, in the first three approximations, for the laws (11),

$$L^{(0)} = 0, \qquad L^{(1)} = \frac{1}{1+a_1}\Big\{\frac{1}{2}(|U^{(0)}_{\eta\eta}|^2)_{\eta\bar\eta} - 2a_2\big[(U^{(0)}_{\eta\eta})^2\big]_{\bar\eta\bar\eta}$$
$$+ \mathrm{Re}\big[(1+2a_1)(U^{(0)}_{\eta\eta}U^{(0)}_{\eta\bar\eta})_{\bar\eta\eta} - 2b_2(U^{(0)}_{\eta\eta}U^{(0)}_{\bar\eta\bar\eta})_{\eta\eta}\big]\Big\},$$

$$L^{(2)} = \frac{1}{1+a_1}\mathrm{Re}\Big\{2b_2\big(U^{(0)}_{\eta\eta}U^{(1)}_{\eta\eta} + U^{(0)}_{\eta\eta}U^{(1)}_{\eta\eta}\big)_{\eta\eta} - (1+2a_1)\big(U^{(0)}_{\eta\eta\eta}U^{(1)}_{\eta\eta}$$
$$+ U^{(0)}_{\eta\eta}U^{(1)}_{\eta\eta\eta}\big)_{\bar\eta} + \big(4a_2U^{(0)}_{\eta\eta}U^{(1)}_{\eta\eta} - U^{(0)}_{\eta\eta}U^{(1)}_{\eta\eta}\big)_{\eta\eta} + \big(4b_3U^{(0)}_{\eta\eta}(U^{(0)}_{\eta\eta})^2$$
$$+ (4b_4 + \tfrac{2}{3})U^{(0)}_{\eta\eta}|U^{(0)}_{\eta\eta}|^2\big)_{\bar\eta\bar\eta} + \big(-4a_1b_2U^{(0)}_{\eta\eta\eta}U^{(0)}_{\eta\eta}U^{(0)}_{\eta\eta} - 4a_2U^{(0)}_{\eta\eta}[(U^{(0)}_{\eta\eta})^2]_{\bar\eta}$$
$$- 4b_2U^{(0)}_{\eta\eta}(U^{(0)}_{\eta\eta}U^{(0)}_{\eta\eta\eta})_{\bar\eta} + 2a_1U^{(0)}_{\eta\eta}(|U^{(0)}_{\eta\eta}|^2)_{\eta} - 2b_2U^{(0)}_{\eta\eta\eta}(U^{(0)}_{\eta\eta}U^{(0)}_{\eta\eta} - U^{(0)}_{\eta\eta}U^{(0)}_{\eta\eta})$$
$$+ (-2b_2U^{(0)}_{\eta\eta}|U^{(0)}_{\eta\eta}|^2 + 4a_3(U^{(0)}_{\eta\eta})^3 + 4a_4U^{(0)}_{\eta\eta}|U^{(0)}_{\eta\eta}|^2\big)_{\eta\eta}\Big\}.$$

The general solution $U^{(k)}$ in each approximation is represented in the form of the sum of a particular solution $U^{*(k)}$ and the known (5) combination of analytic functions

$$2U^{(k)} = 2U^{*(k)} + \eta\overline{\varphi}_k + \bar{\eta}\varphi_k + \chi_k + \overline{\chi}_k.$$

Here the particular solutions are completely determined by the analytic functions of the preceding approximations.

8. As an application, an example is considered of stress concentration around the free contour of a circular hole in a plate under its all-round tension—compression at infinity by the stress p. Omitting the procedure for finding successive approximations, we give the expression for the stress concentration coefficient calculated in three approximations:

$$\chi = \pm 2\left[1 \pm \frac{9Kp}{16(G+3K)G} - \frac{48G^2 + (55+4g_3)GK + (90+12g_3)K^2}{64(G+3K)^2G^2}\,p^2\right],$$

where the upper sign corresponds to tension, the lower to compression.

The concentration coefficient depends on the magnitude and sign of the external load and on the physical constants. Here the first correction to the classical approximation takes into account geometric nonlinearity, and the second also physical nonlinearity through the constant g_3. By applying conformal mappings and the method of N. I. Muskhelishvili, one can solve problems of stress concentration around noncircular contours.

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