ON A CERTAIN PROPERTY OF THE TEICHMÜLLER METRIC

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 517.5

MATHEMATICS

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ON A CERTAIN PROPERTY OF THE TEICHMÜLLER METRIC

(Presented by Academician P. S. Aleksandrov on 20 I 1969)

Let $\mathcal{T}_p$ be the Teichmüller space of marked Riemann surfaces of genus $p > 0$ (see, for example, ([1–3])). The points of this space are closed Riemann surfaces S of genus $p > 0$ with a fixed system of canonical cuts

$$a_1, \dots, a_p; \qquad b_1, \dots, b_p.$$

Let $\omega_1, \dots, \omega_p$ be a basis of Abelian differentials of the first kind on the surface S, normalized by the conditions $\langle \omega_i, a_j \rangle = \delta_{ij}$. The matrix with elements

$$\beta_{ij} = \langle \omega_i, b_j \rangle, \qquad i, j = 1, 2, \dots, p,$$

will be denoted by χS. This matrix is uniquely determined by the surface S and belongs to the Siegel upper half-plane $\mathfrak{Z}_p$ (the space of symmetric complex matrices Z with positive definite imaginary part).

The **Teichmüller distance** $d(S_1, S_2)$ between points $S_1 \in \mathcal{T}_p$ and $S_2 \in \mathcal{T}_p$ is called the infimum of all numbers d for which there exists an e^d-quasiconformal mapping $S_1 \to S_2$ taking the system of canonical cuts of the surface S_1 into the system of canonical cuts of the surface S_2.

It is known (see ([3])) that for $p = 1$ the mapping χ is an isometric mapping of the space $\mathcal{T}_1$ onto the Siegel upper half-plane $\mathfrak{Z}_1$ (which in this case is the ordinary upper half-plane), endowed with the standard non-Euclidean Poincaré metric. For $p > 1$ the analogue of the Poincaré metric is the invariant (with respect to analytic automorphisms) Siegel Riemannian metric ρ, for which

$$ds^2 = \operatorname{Tr}(dZ \cdot Y^{-1} \cdot d\overline{Z} \cdot Y^{-1}),$$

where $Y = \operatorname{Im} Z$. However, since for $p > 1$ the mapping χ is certainly not one-to-one, there is no question of its being isometric. (Moreover, as can be shown without difficulty, for $p > 1$ the mapping χ will not even be locally isometric, and not only with respect to the metric ρ, but also with respect to an arbitrary invariant Finsler metric on the space $\mathfrak{Z}_p$ that is projectively equivalent to the metric ρ.) Nevertheless, it turns out that the metric ρ estimates the metric d from below, i.e., for any points $S_1 \in \mathcal{T}_p$ and $S_2 \in \mathcal{T}_p$ the inequality

$$\rho(\chi S_1, \chi S_2) \leq d(S_1, S_2)$$

holds.

Moreover, an analogous inequality holds for any invariant normalized Finsler metric on the space $\mathfrak{Z}_p$. The maximal one among such metrics is the Finsler metric $\rho_{\max}$, the square of whose line element is equal to the spectral radius of the matrix

$$dZ \cdot Y^{-1} \cdot d\overline{Z} \cdot Y^{-1}.$$

The metric $\rho_{\max}$ is related to the metric ρ by the inequalities

$$\rho \leqslant \rho_{\max} \leqslant \sqrt{p}\,\rho,$$

so that for $p = 1$ both of these metrics coincide.

Thus the following is true.

Theorem. For any points $S_1 \in \mathcal{S}_p$ and $S_2 \in \mathcal{S}_p$ the inequality

$$\rho_{\max}(\varkappa S_1, \varkappa S_2) \leqslant d(S_1, S_2)$$

holds.

It can be shown that this estimate cannot be improved further, i.e., for any $\varepsilon > 0$ there exist distinct points $S_1 \in \mathcal{S}_p$ and $S_2 \in \mathcal{S}_p$ such that

$$\rho_{\max}(\varkappa S_1, \varkappa S_2) \geqslant (1 - \varepsilon) d(S_1, S_2).$$

Before passing to the proof of the theorem just formulated, let us define the notion of the extremal length of a canonical cut.

Let Γ be some family of curves on a Riemann surface S, and let $\sigma = \sigma(z)|dz|$ (where z is a local uniformizing parameter) be a nonnegative conformally invariant piecewise smooth metric given on S. By the symbols $l_\sigma(\gamma)$ and A_σ we shall denote the length of the curve γ and the area of the surface S in the metric σ. By definition,

$$l_\sigma(\gamma) = \int_\gamma \sigma, \qquad A_\sigma = \frac{1}{2i} \iint_S \sigma^2(z)\, dz\, d\bar{z}.$$

Next, put

$$L_\sigma(\Gamma) = \inf_{\gamma\in\Gamma} l_\sigma(\gamma), \qquad \lambda(\Gamma) = \sup_\sigma \frac{L_\sigma^2(\Gamma)}{A_\sigma}.$$

The quantity $\lambda(\Gamma)$ is called the **extremal length of the family** Γ, and a metric σ for which the last supremum is attained is called an **extremal metric**.

By the **extremal length** $\lambda(a)$ **of the canonical cut** a of the Riemann surface S we shall mean the extremal length of the family Γ consisting of curves homotopic to the cut a.

Jenkins' theorem (see [4]) asserts that, for a fairly broad class of families Γ, the extremal metric σ has the form $\sigma = \sqrt{|Q|}$, where $Q = Q(z)(dz)^2$ is a certain quadratic differential on the surface S. In the case that interests us, it is easy to see from Jenkins' proof that this quadratic differential must be the square of some abelian differential of the first kind.

Lemma. For any point S of the Teichmüller space $\mathcal{S}_p$, the extremal length $\lambda(a_1)$ of the canonical cut a_1 is expressed by the formula

$$\lambda(a_1) = 1/\operatorname{Im}\beta_{11}.$$

Proof. According to what was said above, we may restrict ourselves to considering metrics σ of the form

$$\sigma = \left|\sum_{j=1}^{p} c_j\omega_j\right|,$$

where $c_1, \dots, c_p$ are complex constants. In this case

$$A_\sigma = \frac{1}{2i}\iint_S \sum_{j=1}^{p}\sum_{k=1}^{p} c_j\bar{c}_k\omega_j\bar{\omega}_k = \sum_{j=1}^{p}\sum_{k=1}^{p} c_j\bar{c}_k \operatorname{Im}\beta_{jk},$$

and for any curve γ homotopic to the cut a_1,

$$l_\sigma(\gamma) = \int_\gamma \left|\sum_{j=1}^{p} c_j\omega_j\right| \geqslant \left|\int_\gamma \sum_{j=1}^{p} c_j\omega_j\right| = |c_1|.$$

Therefore $L_\sigma(a_1) \geqslant |c_1|$, and

$$\lambda(a_1) \geqslant \max_{c_1,\dots,c_p} \frac{|c_1|^2}{\sum_1^p\sum_1^p c_j\bar{c}_k \operatorname{Im}\beta_{jk}}.$$

Computing this maximum, we obtain $\lambda(a_1) \geqslant (\operatorname{Im}\beta_{11})^{-1}$. To obtain the reverse inequality, note that the geodesics in the metric $\sigma = |\omega_1|$ are the lines $\arg\omega_1 =$ const, and that on these lines

$$\int_\gamma |\omega_1| = \left|\int_\gamma \omega_1\right|.$$

Taking the curve γ_1 to be a geodesic homotopic to the cut a_1, we obtain, consequently, that

$$L_{|\omega_1|}(a_1) \leqslant l_{|\omega_1|}(\gamma_1) = \left|\int_{\gamma_1} \omega_1\right| = 1,$$

and therefore

$$\lambda(a_1) \geqslant 1/A_{|\omega_1|} = 1/\operatorname{Im}\beta_{11}.$$

Thus the lemma is completely proved.

Furthermore, it is known (see, for example, ([1],[3])) that the extremal lengths of families of curves possess the property of quasi-invariance under quasiconformal mappings. In our case this means that, under any K-quasiconformal mapping $S \to S'$ for which the canonical cut a passes into the canonical cut a', the inequality $\lambda(a') \leqslant K\lambda(a)$ holds. Hence, from the lemma proved above, the following immediately follows.

Corollary. *For any two points S_1 and S_2 of Teichmüller space that are at distance $d = d(S_1, S_2)$, the inequality*

$$\operatorname{Im}(\chi S_2)_{11} \leqslant e^d \operatorname{Im}(\chi S_1)_{11}$$

holds (by the symbol A_{ij} we denote here the element of the matrix A located at the intersection of the i-th row and the j-th column).

Let us now proceed to the proof of the theorem. As is known (see, for example, ([1])), the automorphism group of the fundamental group of Riemann surfaces of genus p acts isometrically on the space $\mathcal{S}_p$. Therefore, according to the corollary, for any such automorphism φ the inequality

$$\operatorname{Im}(\chi(\varphi S_2))_{11} \leqslant e^d \operatorname{Im}(\chi(\varphi S_1))_{11}. \tag{1}$$

will hold.

On the other hand, it is clear that each automorphism φ determines a certain modular automorphism of the Siegel upper half-plane $\mathfrak{Z}_p$ (we shall denote it by the same symbol φ); moreover, for any Riemann surface $S \in \mathcal{S}_p$ the equality

$$\chi(\varphi S) = \varphi(\chi S)$$

holds.

According to the approximation theorem proved by the authors of this note (see ([5])), the automorphism φ can be chosen so that the matrix $\chi(\varphi S_1)$ has the form $(X + iE + U)\mu$, while the matrix $\chi(\varphi S_2)$ has the form $(X + iT + V)\mu$, where μ is a certain positive number; X is a certain real matrix; U and V are matrices (generally speaking, complex) all of whose elements do not exceed in absolute value an arbitrary preassigned number $\delta > 0$, and T is a diagonal matrix $\operatorname{diag}(T_{11}, \dots, T_{pp})$, where $1 \leqslant T_{pp} \leqslant \cdots \leqslant T_{11}$. Here the matrix T is determined only by the surfaces S_1 and S_2 (does not depend on the automorphism φ), and the quantity $\ln T_{11}$ is equal to the distance between the matrices χS_1 and χS_2 in the metric $\rho_{\max}$. Thus, for this automorphism φ, inequality (1) can be rewritten in the form $\rho_{\max} \leqslant d + \varepsilon$, where $\varepsilon \to 0$ as $\delta \to 0$. Consequently, $\rho_{\max} \geqslant d$, as was asserted.

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Received
9 I 1969

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Note: Figure translations are in progress. See original paper for figures.