A STUDY, BY SPECTRAL CRITERIA, OF POWER STABILITY OF DIFFERENCE APPROXIMATIONS OF PROBLEMS WITH TIME

1969

SovietRxiv

Abstract

Full Text

MATHEMATICS

LE DINH THINH

A STUDY, BY SPECTRAL CRITERIA, OF POWER STABILITY OF DIFFERENCE APPROXIMATIONS OF PROBLEMS WITH TIME

(Presented by Academician A. N. Tikhonov, 29 IV 1969)

Any approximation of a mixed problem for a system of differential equations in two variables (t, x) with constant coefficients can be written in the form (^{1})

$$\sum_{k=0}^{1} A_k^1 u_{n+k}^{m+1} + \sum_{k=0}^{1} A_k^0 u_{n+k}^{m} = 0, \tag{1}$$

$$\sum_{i=0}^{1} B_i u_0^{m+i} = 0, \tag{2}$$

$$\sum_{i=0}^{1} C_i u_N^{m+i} = 0. \tag{3}$$

Here $A_k^i,\ i, k = 0, 1$, are square matrices of dimension $l \times l$; $B_i,\ i = 0, 1$, are rectangular matrices of dimension $l_1 \times l$; $C_i,\ i = 0, 1$, are rectangular matrices of dimension $l_2 \times l$. By u_n^m is denoted the value of the solution of the grid problem at the node (m, n), the point $(m\tau, nh)$, where τ and h are the mesh sizes in t and x, respectively.

In what follows, without further stipulation, we assume that $l_1 + l_2 = l$; it is possible here that either $l_1 = 0$ or $l_2 = 0$.

Under certain natural solvability conditions, problems (1)–(3) can be represented (^{2,3}) in the form $\mathbf{u}^{m+1} = R_N \mathbf{u}^m$; here R_N is a certain linear operator, $\mathbf{u}^m = \{u_0^m, \dots, u_N^m\}$.

Suppose that in the space U of vectors $\mathbf{u}^m$ of this form a norm is introduced. We shall assume that it is one of the l_p norms:

$$\|\mathbf{u}^m\| = \left(\frac{1}{N+1} \sum_{n=0}^{N} \|u_n^m\|_{(2)}^p \right)^{1/p},$$

where, in turn, $\|u_n^m\|_{(2)}$ is the Euclidean norm in the space of vectors of dimension l.

In (2,3) the notion of the spectrum of a family of operators $\{R_N\}$ was introduced:

Definition. Let the complex number z satisfy the condition: for any $\varepsilon > 0$ one can specify $N_0(\varepsilon, z)$ such that for every $N > N_0(\varepsilon, z)$ there exists a vector u satisfying the inequality $\|R_{Nu} - zu\| \leqslant \varepsilon\|u\|$. The totality of all such z will be called the **spectrum Λ of the family of operators** $\{R_N\}$.

There it was also indicated that the spectrum has the representation $\Lambda = \Lambda_x \cup \Lambda_y \cup \Lambda_z$ and an algorithm for constructing the sets $\Lambda_x, \Lambda_y, \Lambda_z$.

Let Λ_y be the set of those z for which the grid problem

$$\sum_{i=0}^{1}(A_i^1 z + A_i^0)\, v_{n+i} = 0, \qquad -\infty < n < \infty,$$

has a bounded solution; $\overline{\Lambda}_x$ is the set of those z for which the mesh problem has a bounded solution

$$\sum_{i=0}^{1}(A_i^1 z + A_i^0)\mathbf{v}_{n,i} = 0 \quad (B_1 z + B_0)\mathbf{v}_0 = 0, \quad n \geq 0;$$

$\overline{\Lambda}_z$ is the set of those z for which the mesh problem has a bounded solution

$$\sum_{i=0}^{1}(A_i^1 z + A_i^0)\mathbf{v}_{n+i} = 0 \quad (C_1 z + C_0)\mathbf{v}_N = 0, \quad n < N.$$

By $\Lambda_x, \Lambda_y, \Lambda_z$ we shall denote the closures of the sets $\overline{\Lambda}_x, \overline{\Lambda}_y, \overline{\Lambda}_z$. The indicated sets do not depend on N and consist of a finite number of points and analytic arcs in the complex z-plane; moreover $\overline{\Lambda}_y = \Lambda_y$.

In (2, 3) a necessary condition for uniform boundedness of the powers of the operators R_N was obtained: in order that the quantities $\|(R_N)^m\|$ be bounded uniformly in m and N, it is necessary that the condition $\Lambda \subseteq K_1$ be satisfied; here and below K_1 is the disk $|z| \leq 1$, S_ρ is the circle $|z| = \rho$. This condition is at present regarded as a necessary condition for the practical suitability of a mesh approximation.

Below we construct an example showing that the condition $\Lambda \subseteq K_1$ is not a sufficient condition for stability. We shall consider a notion weaker than stability, namely the notion of power stability.

Definition. The mesh problem (1)–(3) has **power stability** for $m \leq \varphi(N)$ if, for some c, γ, r, when $m \leq \varphi(N)$ the relation

$$\|u^m\| \leq c m^{\gamma} N^r \|u^0\|$$

holds.

The function

$$\mathbf{v}_n(z) = \sum_{m=0}^{\infty} \mathbf{u}_n^m z^{-m}$$

is analytic in the domain $|z| > \|R_N\|$, in consequence of the inequality $\|u^m\| \leq \|R_N\|^m \|u^0\|$. Continue it analytically for $|z| \leq \|R_N\|$; we can write the equality

$$\mathbf{u}_n^m = \frac{1}{2\pi i} \oint_{\Gamma} \mathbf{v}_n(z) z^{m-1}\, dz; \tag{4}$$

here Γ is any contour that can be obtained by a continuous deformation of the contour $|z| = \|R_N\| + \varepsilon$, where $\varepsilon > 0$, without passing through the singular points of the function $\mathbf{v}_n(z)$.

The function $\mathbf{v}_n(z)$ satisfies the mesh problem

$$A_0(z)\mathbf{v}_n + A_1(z)\mathbf{v}_{n+1} = \mathbf{f}_n, \quad n = 0, \dots, N-1; \quad B(z)\mathbf{v}_0 = \varphi; \quad C(z)\mathbf{v}_N = \psi,$$

where $A_i(z) = A_i^1 z + A_i^0,\ i = 0, 1;\ B(z) = B_1 z + B_0;\ C(z) = C_1 z + C_0$.

We shall call the equation

$$|A_0(z) + \lambda A_1(z)| = 0. \tag{6}$$

the characteristic equation of the system (1)–(3).

Definition. The roots $\lambda_j(z)$ of equation (6) will be called **generalized eigenvalues**; if $\det A_1(z) = 0$, then equation (6) has $q < l$ roots (counting their multiplicities). In this case we assume that equation (6) has $l - q$ roots

$$\lambda_{q+1}(z) = \dots = \lambda_l(z) = \infty.$$

To each root $\lambda_k(z)$ we associate a normalized eigenvector or associated vector of the matrix $A_0(z) + \lambda A_1(z)$, corresponding to its finite generalized eigenvalue. To a generalized eigenvalue $\lambda_k(z) = \infty$ we associate eigenvectors or associated vectors of the matrix $A_1(z)$, corresponding to its zero eigenvalue.

We shall establish this correspondence so that the following conditions are satisfied: 1) for any z, each of the eigenvectors and associated vectors corresponds to some eigenvalue; in other words, for each z the system of vectors $\{C_k(z)\}$ is complete; 2) $\lambda_k(z)$ and the vectors $C_k(z)$ are analytic functions of z for $|z| < \infty$, with the exception of a finite number of points $P_1, \dots, P_s$; 3) $\lambda_j(z)$ are continuous functions of z at the points P_j. We shall choose the contour of integration so that it does not pass through these points.

If the root $\lambda(z)$ of multiplicity s corresponds to s linearly independent eigenvectors, then we say that this root has a **simple structure**.

In what follows, unless otherwise stated, we assume $\Lambda \subseteq K_1$. In this case, for $|z| > 1$, equation (6) has l_2 generalized eigenvalues greater than 1 in modulus and l_1 smaller than 1 in modulus; we renumber the generalized eigenvalues so that, for $|z| > 1$, one has $|\lambda_1(z)|, \dots, |\lambda_{l_1}(z)| < 1$.

Definition. The set of points $z \in S_1$ satisfying the condition

$$\inf_{i\le l_1,\, j>l_1} \left(|\lambda_j(z)| - |\lambda_i(z)|\right) = \delta(z) > 0 \tag{7}$$

will be denoted by S'.

Definition. Denote by $\overline{\Lambda}_y^{\sigma}$ the set of points $z_0 \in \Lambda_y \cap S_1$ satisfying the condition: there exist $\gamma, C_1 > 0$ such that

$$|\lambda_j(z)| - |\lambda_i(z)| \ge C_1 |z - z_0|^{\sigma}$$

for $i \le l_1 < j,\ |z - z_0| \le \gamma,\ 1 < |z| < 1 + \gamma$.

Definition. Put

$$\Lambda_y^{\sigma} = \bigcup_{\sigma' < \sigma} \overline{\Lambda}_y^{\sigma'}.$$

Let $A_1(z), A_2(z)$ be matrices of dimensions $l \times l_1,\ l \times l_2$, respectively; the columns of $A_1(z)$ are the vectors $C_1(z), \dots, C_{l_1}(z)$, and the columns of $A_2(z)$ are the vectors $C_{l_1+1}(z), \dots, C_l(z)$. By $A_1^{ij}(z)$, for $i \le l_1 \le j$, we denote the matrix obtained from $A_1(z)$ by replacing the i-th column by the vector $C_j(z)$.

Definition. Denote by Λ_{xy}^{σ} the set of points $z_0 \in \Lambda_y \cap S_1$ satisfying the following conditions: there exist $C_1, C_2, C_3, \gamma > 0$ such that, for $|z - z_0| \le \gamma,\ 1 < |z| < 1 + \gamma$:

$$1) \quad C_1 \left(\inf_{j > l_1} |\lambda_j(z)| - \sup_{i \le l_1} |\lambda_i(z)| \right) |z - z_0|^{-\sigma} \le C_2,$$

$$2) \quad \left| \frac{\det(B(z) A_1^{ij}(z))}{\det(B(z) A_1(z))} \right| \le C_3$$

for all $i \le l_1,\ j > l_1$ such that

$$|\lambda_j(z)| - |\lambda_i(z)| = O(|z - z_0|^{\sigma}).$$

The set Λ_{yz}^{σ} is defined analogously.

The proof of the following theorems is carried out as follows: we write an explicit expression for the Green' s function of the grid problem (5) and an expression for v_n in terms of the Green' s function. It turns out that the Green' s function is analytic at those points where the determinant

$$\Delta_N(z) = \begin{vmatrix} B(z)((A_1(z),\, A_2(z))) \\ C(z)((A_1(z),\, A_2(z))D(\lambda^N)) \end{vmatrix}, \qquad \text{where} \qquad D(\lambda^N) = \begin{pmatrix} \lambda_1^N & 0 & \\ 0 & \ddots & \lambda_l^N \end{pmatrix}.$$

does not vanish.

Next, depending on the properties of the determinant, it is shown that outside some neighborhood S_ρ, $\rho_N > 1$, one has $\Delta_N \neq 0$. Then, in (4), the contour of integration $\Gamma = S_{1+\omega(\rho_N - 1)}$ is chosen for some finite $\omega > 1$, and the integral is estimated.

Theorem 1. Suppose: 1) $\Lambda \subseteq K_1$; 2) $(\Lambda \cap S_1) \subseteq S'$.

Then

$$\rho_N \geq 1 + C_1 q^N, \qquad \text{where } C_1 > 0, \quad 0 < q < 1,$$

and the stability estimate has the form

$$\|u^m\| \leq C_2 m^\nu N^r \exp(nC_3 q^N)\|u^0\|.$$

Theorem 2. Suppose: 1) $\Lambda \subseteq K_1$; 2)

$$(\Lambda \cap S_1) \subseteq (S' \cup (\Lambda_{xy}^\sigma \cap \Lambda_{yz}^\sigma) \cup (\Lambda_{xy}^\sigma \cap \Lambda_y^\sigma) \cup (\Lambda_{yz}^\sigma \cap \Lambda_y^\sigma));$$

3) the roots $\lambda_j(z)$ of equation (6), different from $\lambda = 0$ and $\lambda = \infty$, have simple structure everywhere except for a finite number of points.

Then

$$\rho_N \geq 1 + C_4 N^{-1/\sigma}, \qquad C_1 > 0,$$

and the stability estimate has the form

$$\|u^m\| \leq C_5 m^\nu N^r \exp\left(C_6 m/N^{1/\sigma}\right)\|u^0\|.$$

Theorem 3. Suppose: 1) $\Lambda \subseteq K_1$; 2)

$$(\Lambda \cap S_1) \subseteq (S' \cup \overline{\Lambda}_y^\sigma).$$

Then

$$\rho_N \geq 1 + C(\ln N/N)^{1/\sigma}, \qquad C > 0.$$

The stability estimate has the form

$$\|u^m\| \leq C_6 m^\nu N^r \exp\left(C_7 m/(N/\ln N)^{1/\sigma}\right)\|u^0\|.$$

Suppose we have an approximation of a mixed problem in the domain $0 \leq t$, $0 \leq x \leq 1$ for the hyperbolic system $\mathbf{u}_t = A\mathbf{u}_x$ with boundary-condition operators independent of time. Then this approximation will have the form (1)–(3), i.e., its coefficients do not depend on N, if, as $\tau, h \to 0$, the condition $\tau/h = \varkappa = \text{const}$ is satisfied. Similarly, in the case of an approximation of the parabolic system $\mathbf{u}_t = A\mathbf{u}_{xx}$, the condition $\tau/h^2 = \varkappa = \text{const}$ must be satisfied. Therefore, in the first case, integration over the time interval $0 \leq t \leq T$ corresponds to $m \leq (T/\varkappa)N$, and in the second case to $m \leq (T/\varkappa)N^2$.

For approximations of parabolic systems in the case $\sigma = 1/2$, the estimate of Theorem 3 does not ensure power stability when $\varphi(N) = CN^2$, and consequently is of no interest. For approximations of hyperbolic systems in the case $\sigma = 1$, the estimate of Theorem 3 for $m = (T/\varkappa)N$ becomes the estimate

$$\|u^m\| \leq C_6(T)N^{C_7+C_8T}\|u^0\|, \tag{8}$$

which also is unlikely to be of practical interest.

Lemma. If $\Delta_N(z) = 0$, then $\|R_N^m\| > |z|^m$.

It can be shown that for problem (1)–(3), having the form

$$u_n^{m+1} - u_n^m + \frac{r}{2}(u_{n+1}^{m+1} - u_{n-1}^{m+1}) = 0, \quad u_1^{m+1} - u_0^{m+1} = 0, \quad u_N^{m+1} = 0$$

(note that this is an approximation of the equation $u_t + u_x = 0$), the conditions of Theorem 3 are satisfied for $\sigma = 1$, and $\Delta_N(z_0) = 0$ for $|z_0| \geq 1 + C\ln N/N$. Hence it follows that, for some initial condition u^0, $\|u^m\| \geq N^{CT}\|u^0\|$, $C > 0$, i.e., estimate (8) cannot be substantially improved.

This example shows that problems (1)–(3) satisfying the condition $\Lambda \subseteq K_1$ may have error growth that makes them unsuitable for actual computations; that is, investigation of approximations only by means of this condition, without invoking theorems of type (1)–(3), is insufficient. A more detailed exposition of the main results of the work is given in [5].

I express my deep gratitude to N. S. Bakhvalov for his comprehensive assistance in carrying out this work.

Received23 IV 1969

CITED LITERATURE

[1] S. K. Godunov, V. S. Ryabenkii, *Zhurnal vychislitel' noi matematiki i matematicheskoi fiziki*, **3**, No. 2, 211 (1963).
[2] S. K. Godunov, V. S. Ryabenkii, *Introduction to the Theory of Difference Schemes*, Moscow, 1963.
[3] S. K. Godunov, V. S. Ryabenkii, *Uspekhi matematicheskikh nauk*, **18**, issue 3 (111), 3 (1963).
[4] V. S. Ryabenkii, A. F. Filippov, *On the Stability of Difference Equations*,

Moscow, 1956.
[5] Le Dinh Thinh, Candidate' s Dissertation, Moscow University, 1969.

Note: Figure translations are in progress. See original paper for figures.