ON THE DIRICHLET PROBLEM FOR EQUATIONS WITH COEFFICIENTS BELONGING TO SPACES WITH MIXED NORM

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Abstract

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ON THE DIRICHLET PROBLEM FOR EQUATIONS WITH COEFFICIENTS BELONGING TO SPACES WITH MIXED NORM

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Known results ([3], [4–6]) concern the first boundary-value problem for the uniformly elliptic equation

$$Lu \equiv \frac{\partial}{\partial x_i}\left(a_{ij}u_{x_j} + a_i u\right) + b_i u_{x_i} + au = \frac{\partial f_i}{\partial x_i} + f \qquad \text{(I)}$$

with discontinuous unbounded lower-order coefficients. In particular, in ([2]) conditions are given for the existence and uniqueness of a generalized solution from W_2^1 of the Dirichlet problem for equation (I). These conditions are, in a certain sense, not only sufficient but also necessary, i.e., in terms of L_p-spaces these results cannot be improved.

However, equations are known a priori whose coefficients do not satisfy the indicated conditions, but for which existence and uniqueness theorems for the Dirichlet problem hold. The present paper is devoted to the study of generalized solutions of the Dirichlet problem for such equations. Existence and uniqueness theorems are obtained for the solution of the Dirichlet problem for equations of the form (I) with coefficients belonging to a space with mixed norm. In addition, conditions for boundedness of the generalized solution are given. The results obtained cannot be improved in terms of the spaces under consideration. The paper uses the results of ([2], [3], [6]).

Let s be a natural number not exceeding n; E^n is n-dimensional Euclidean space. We shall denote each point $x \in E^n$ in the form of a pair of vectors (x_s, x_{n-s}) or $(x^{(1)}, x^{(2)})$, where

$$x_s(x_1, \dots, x_s) \equiv x^{(1)}(x_1^{(1)}, \dots, x_{n_1}^{(1)}), \quad x_{n-s}(x_{s+1}, \dots, x_n) \equiv x^{(2)}(x_1^{(2)}, \dots, x_{n_2}^{(2)}),$$

$n_1 = s,\ n_2 = n - s$. Let E^{n_i} be the n_i-dimensional space of vectors $x^{(i)}$ $(i = 1, 2)$.

By $|x^{(i)} - y^{(i)}|$ we shall denote the distance between the points $x^{(i)}$ and $y^{(i)}$ in E^{n_i}; $|x-y|$ is the distance between x and y in E^n.

Let D be a bounded domain in E^n and $D_1 = D \cap (x^{(2)} = \text{const})$; $D_2 = \operatorname{pr}_{E^{n-s}} D$. Denote by $L_{(p_1,p_2)}(D_1, D_2)$ the set of functions defined in D and satisfying the condition

$$\Big\| \|f(x)\|_{L_{p_1}(D_1)} \Big\|_{L_{p_2}(D_2)} < \infty,$$

where

$$\|\cdot\|_{L_{p_i}(D_i)} = \begin{cases} \left(\displaystyle\int_{D_i} |\cdot|^{p_i}\, dx^{(i)} \right)^{1/p_i}, & \text{if } 1 \le p_i < \infty, \\ \operatorname*{vrai\,max}_{x^{(i)} \in D_i} |\cdot|, & \text{if } p_i = \infty. \end{cases}$$

If in $L_{(p_1,p_2)}(D_1, D_2)$ one introduces the norm by the equality

$$\|f\|_{L_{(p_1,p_2)}(D_1,D_2)} = \Big\| \|f\|_{L_{p_1}(D_1)} \Big\|_{L_{p_2}(D_2)},$$

then $L_{(p_1,p_2)}(D_1, D_2)$ will be a complete normed space.

In the case when we are dealing with only one space $L_{(p_1,p_2)}(D_1, D_2)$, one may use a more convenient notation for them, namely

$$D_{(p_1,p_2)}(D_1, D_2) \equiv L_{(p_1,p_2)}(D) \equiv L_p(D).$$

Let a be any nonnegative number; k a positive integer; β a fixed positive number. Denote by Y_β the following class of Banach spaces:

$$Y_\beta = \{L_{(\beta p, p)}(D_1, D_2);\ \max\{1/\beta, 1\} < p < \infty\}.$$

For $\beta \ne 1$, Y_β forms a continuous scale, different from the scale of L_p spaces. Obviously,

$$Y_1 = \{L_p\}$$

($\{L_p\}$ is the scale of L_p-spaces). By Y we denote the set of scales Y_β for all possible nonnegative values of β, i.e.

$$Y = \{Y_\beta;\ 0 < \beta < \infty\}.$$

From each scale Y we consider one representative, and the set of these representatives will be denoted by X_k^α. The class X_k^α is defined as follows:

$$X_k^\alpha = \begin{cases} L_{(r_1,r_2)}(D_1, D_2); \quad kr_1r_2 - (n-s)r_1 - sr_2 - \alpha = 0; \\ \infty > r_1 > \begin{cases} 1, & \text{if } 1 < s < k, \\ s/k, & \text{if } s \geq k; \end{cases} \\ \infty > r_2 > \begin{cases} 1, & \text{if } n-s < k, \\ (n-s)/k, & \text{if } n-s \geq k. \end{cases} \end{cases}$$

Denote by $\Omega_k^{(\alpha)}$ the set of those points (r_1, r_2) of the plane r_1Or_2 for which

$$L_{(r_1,r_2)}(D_1, D_2) \in X_k^\alpha.$$

Let p be an integer greater than or equal to one. The class $X_{k;p}^\alpha$ is defined by the equality

$$X_{k;p}^\alpha = \{L_{(r_1,r_2)}(D_1, D_2);\ (r_1, r_2) \in \Omega_k^{(\alpha)} \cap \{p \leq r_1, r_2\}\}.$$

Put

$$\Omega_{k;p}^{(\alpha)} = \Omega_k^{(\alpha)} \cap \{p \leq r_1, r_2\}.$$

Theorem 1. If D is a bounded domain of n-dimensional Euclidean space and

$$s/p_1 + (n-s)/p_2 - 1 \leq s/q_1 + (n-s)/q_2; \qquad 1 \leq p_2 \leq p_1 \leq q_1, q_2,$$

then for every function $f \in \mathring{W}^1_{(p_1,p_2)}(D)$ the estimate

$$\|f\|_{L_{(q_1,q_2)}(D)} \leq C_1 \|\nabla f\|^\sigma_{L_{(p_1,p_2)}(D)} \|f\|^{1-\sigma}_{L_{(p_1,p_2)}(D)},$$

holds, where C_1 is a constant independent of D, and σ is determined from the equality

$$\sigma = s(1/p_1 - 1/q_1) + (n-s)(1/p_2 - 1/q_2).$$

Theorem 2. If D is a bounded domain of n-dimensions, star-shaped with respect to some ball, and $f \in W_2^1(D)$, then, for

$$n/2 - 1 \leq s/p_1 + (n-s)/p_2, \qquad 2 \leq p_1, p_2,$$

the function f belongs to the space $L_{(p_1,p_2)}(D)$, and, moreover, the estimate

$$\|f\|_{L_{(p_1,p_2)}(D)} \leq C_2 \|f\|_{W_2^1(D)}.$$

Theorem 3. Let the positive numbers α, p_1, p_2 be such that

$$\alpha > \max\{1 - 2/p_1,\ 1 - 2/p_2\}, \qquad s/p_1 + (n-s)/p_2 = n/2 - \alpha;$$

$$2 \leq p_1, p_2,$$

and let the function $f \in W_2'(D)$ satisfy the condition

$$\int_D f(x)\, dx = 0;$$

then

$$\|f\|_{L_{(p_1,p_2)}(D)} \leq C_2 \|\nabla f\|^{\alpha}_{L_2(D)} \|f\|^{1-\alpha}_{L_2(D)},$$

where the constant C_2 depends on the domain D, but remains unchanged under a similarity transformation.

Theorem 4. If the numbers α, p_1, p_2 satisfy the conditions

$$\alpha > \max\{1 - 2/p_1,\ 1 - 2/p_2\}, \qquad s/p_1 + (n-s)/p_2 = n/2 - \alpha,$$

$$2 \leq p_1, p_2,$$

then for any function f in $W_2^1(D)$ the estimate

$$\|f\|_{L_{(p_1,p_2)}(D)} \leq C_3 (\|f\|_{L_2(D)} + \|\nabla f\|^{\alpha}_{L_2(D)} \|f\|^{1-\alpha}_{L_2(D)}).$$

holds.

Theorem 5. If D is a bounded domain of n-dimensional Euclidean space and $f \in \dot{W}_2^1(D)$, then the estimate

$$\|f\|_{L_2(D)} \leq C_1 \bigl[\operatorname{mes}(\operatorname{pr}_{E^{n-s}} D)\bigr]^{1/(n-s)} \|\nabla f\|_{L_2(D)}$$

holds, where C_1 is the constant from Theorem 1.

Theorem 6. If the function $f \in W_2^1(D)$ has a bounded $\operatorname{vrai\,max}_S f(x)$ and, for $k \geq k_0 \geq \operatorname{vrai\,max}_S f(x)$, satisfies the inequalities

$$\int_{A_k} |\nabla f|^2\, dx \leq \gamma \left[\int_{A_k} (f-k)^2\, dx + k^2 \|1\|^2_{L_{(p_1,p_2)}(A_k)} \right], \tag{1}$$

where

$$s/p_1 + (n-s)/p_2 = (n/2 - 1)(1 + \varepsilon), \qquad \varepsilon > 0, \tag{2}$$

then $\operatorname{vrai\,max}_D f(x)$ is bounded.

Theorem 7. Let D be an n-dimensional bounded domain with piecewise smooth boundary S, and let the function $f \in W_2^1(D)$, for $k \geq \tilde{k}$, satisfy inequalities (1), (2).

Then $\operatorname{vrai\,max}_D f(x)$ is estimated by a constant depending on $\tilde{k}, C_1, \gamma, \varepsilon, n, \|f\|_{L_1(A_{\tilde{k}})}$, and on the boundary S.

Theorem 8. Let $f \in W_2^1(K_{(1+\delta)R})$, $K_{(1+\delta)R} \subset D$, $\delta > 0$, and suppose that for any pair of balls $K_{(1-\sigma)\rho}$ and K_ρ, concentric with $K_{(1+\delta)R}$, with $\widetilde{R} \leq \rho(1-\sigma) < \rho \leq (1+\delta)R$, the following holds:

$$\int_{A_{k,\rho(1-\sigma)}} |\nabla f|^2 \, dx \leq \gamma \left[(\sigma\rho)^{-2} \int_{A_{k,\rho}} (f-k)^2 \, dx + \rho^{-2r} k^2 \|1\|^2_{L_{(p_1,p_2)}(A_{k,\rho})} \right],$$

where $\delta, \gamma, k, p_1, p_2, r$ are fixed positive numbers, and

$$s/p_1 + (n-s)/p_2 = n/2 - 1 + r.$$

Then $\operatorname{vrai\,max}_{K_R}$ does not exceed a certain number determined only by

$$k, \delta, n, r, \gamma, p_1, p_2 \quad \text{and} \quad R^{-n} \int_{K_{(1+\delta)R}} |f(x)|^2 \, dx;$$

$A_{k,\rho}$ is the set of points x in K_ρ for which $f(x) > k$.

Let us now consider, in a bounded domain $D \subset E^n$ with boundary S, the equation

$$Lu \equiv \frac{\partial}{\partial x_i}(a_{ij} u_{x_j} + a_i u) + b_i u_{x_i} + au = \frac{\partial f_i}{\partial x_i} + f \tag{3}$$

with coefficients satisfying the conditions:

$$\nu \xi_i \xi_i \leq a_{ij} \xi_i \xi_j \leq \mu \xi_i \xi_i, \qquad \nu, \mu = \text{const} > 0; \tag{4}$$

$$\|a_i^2;\, b_i^2;\, a\|_{L_{(r_1,r_2)}(D_1,D_2)} \leq K; \tag{5}$$

$$\left\|\sum f_i^2\right\|_{L_1(D)}, \qquad \|f\|_{L_{(q_1,q_2)}(D_1,D_2)} < \infty; \tag{6}$$

$$(r_1, r_2) \in \Omega_2^{(\alpha)}, \qquad (q_1, q_2) \in \Omega^{(0)}_{n/2+1}. \tag{7}$$

Theorem 9. The Dirichlet problem in $D' \subset D$ for equation (3) has no more than one generalized solution from W_2^1, if conditions (4), (5), (7) are satisfied and $\mathrm{mes}(\mathrm{pr}_{E^n} D')$ is sufficiently small.

Theorem 10. The Dirichlet problem for equation (3) in domains $D' \subset D$ of arbitrary size has no more than one generalized solution from W_2^1, provided that conditions (4), (5), (7) are satisfied and $a(x) < -N$, where N is a sufficiently large positive number.

Theorem 11. Let $(r_1, r_2) \in \Omega_2^{(\alpha)}$, $(q_1, q_2) \in \Omega_{n/2+1}^{(0)}$, and suppose that conditions (4), (5), (6), (7) are satisfied and $a(x) < -N$; then the Dirichlet problem for equation (3) has a generalized solution $u(x)$ from W_2^1 for any boundary value $\varphi(x)$ from W_2^1.

Theorem 12. Let $(r_1, r_2) \in \Omega_2^{(\alpha)}$ and suppose that the conditions

$$\nu\xi_i\xi_i \leqslant a_{ij}\xi_i\xi_j \leqslant \mu\xi_i\xi_i, \qquad \nu, \mu = \mathrm{const} > 0, \qquad \|a_i^2, b_i^2, f_i^2, a, f\|_{L_{(r_1,r_2)}(D)} \leqslant K.$$

are satisfied. Then for any generalized solution $u(x)$ from W_2^1 of equation (1) in any subdomain D' of the domain D, the quantity $\mathrm{vrai\,max}_{D'} u(x)$ is finite, i.e.

$$\mathrm{vrai\,max}_{D'} |u(x)| < C,$$

where C is a constant depending only on $\nu, \mu, K, r_1, r_2, s, n, \|u\|_{L_2(D)}$ and the distance of D' from the boundary S of the domain D.

Theorem 13. If the conditions of the preceding theorem are satisfied and

$$\mathrm{vrai\,max}_S |u(x)| < M,$$

then

$$\mathrm{vrai\,max}_D |u(x)|$$

is bounded.

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REFERENCES

1. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Publishing House of the Siberian Branch of the Academy of Sciences of the USSR, 1962.

2. O. A. Ladyzhenskaya, N. N. Ural' tseva, *Linear and Quasilinear Equations of Elliptic Type*, Nauka, 1964.

3. O. A. Ladyzhenskaya, N. N. Ural' tseva, Trudy Mat. Inst. im. V. A. Steklova AN SSSR, 23, 172 (1962).

4. C. Miranda, Ann. matem., 61, 151 (1963).

5. G. Stampacchia, Ann. Inst. Fourier, 15, 1, 189 (1965).

6. C. B. Morrey, Math. Zs., 72, 146 (1959).

7. A. D. Aleksandrov, Vestn. LGU, No. 7 (1967).

8. O. A. Oleinik, DAN, 124, No. 6, 1219 (1959).

9. V. P. Il' in, Trudy Mat. Inst. im. V. A. Steklova AN SSSR, 26, 227 (1962).

Note: Figure translations are in progress. See original paper for figures.