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Abstract

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GAUSSIAN STATIONARY PROCESSES WITH ASYMPTOTICALLY POWER-LAW SPECTRUM

(Presented by Academician A. N. Kolmogorov on 21 V 1968)

1. Let $\xi(\varphi)$, $\varphi(t) \in K_T$, be a real Gaussian process on the interval $[0, T]$, generalized in the sense of Gelfand–Ito, with zero mean and correlation functional $\bar{B}(\varphi, \psi) = M\xi(\varphi)\xi(\psi)$; K_T is the space of infinitely differentiable functions $\varphi(t)$ with support in $[0, T]$ and with the usual topology of L. Schwartz.

Denote by $H_t(\xi)$, $t \leq T$, the Hilbert spaces obtained by closing the linear span of the random variables $\{\xi(\varphi), \varphi \in K_t\}$ with respect to the scalar product $(\xi(\varphi), \xi(\psi)) = B(\varphi, \psi)$. The space $H_T(\xi)$ is isomorphic to the space of functionals $H_T(B) \subset K'_T$, called the space with reproducing kernel $B(\varphi, \psi)$ (see ([2])). The isomorphism $H_T(\xi) \leftrightarrow H_T(B)$ is effected by the unitary correspondence $U:\ H_T(\xi) \ni \eta \leftrightarrow (U\eta)(\varphi) = M\eta\xi(\varphi) \in H_T(B)$ and

$$\langle f_1, f_2 \rangle_{H_T(B)} = (U^{-1}f_1, U^{-1}f_2),$$

$f_1, f_2 \in H_T(B)$.

A number of statistical problems, such as extrapolation of the process $\xi(\varphi)$, description of measures equivalent to the measure induced by the process $\xi(\varphi)$, etc., are solved by using the spaces $H_T(\xi)$ and $H_T(B)$ (see, for example, ([2–5])). In the present note a method is developed for the analytic description of the spaces $H_T(\xi)$ and $H_T(B)$ on the basis of the canonical representation ([6]) of the process $\xi(\varphi)$ and the solution of certain Wiener–Hopf type equations by a method close to that used by M. G. Krein ([7]) in inverse problems for an inhomogeneous string. This makes it possible to study $H_T(\xi)$ and $H_T(B)$ for processes $\xi(\varphi)$ that include, as particular cases, the Gaussian process $\xi(t)$ with stationary increments

$$M|\xi(t) - \xi(s)|^2 = \text{const} \cdot |t - s|^\alpha, \quad \alpha > 0,$$

and stationary processes with spectral density

$$c_1 < f(\lambda)(1+|\lambda|)^\mu < c_2, \quad |\mu| < \infty.$$

A somewhat different approach is used in Sec. 5 for the study of homogeneous fields.

2. We note a simple property of the functionals from $H_T(B)$. Let $X \supset K_T$ be a linear topological space containing K_T as an everywhere dense subset, and suppose the topology in K_T induced from X is weaker than the topology of K_T; we shall call X an extension of K_T. If the correlation functional $B(\varphi, \varphi)$ is continuous in the extension $X \supset K_T$, then $H_T(B) \subset X'$, where X' is the conjugate space to X. This follows from the Cauchy –Bunyakovsky inequality:

$$|f(\varphi)|^2 = |(U^{-1}f, \xi(\varphi))|^2 \leq \|U^{-1}f\|^2 B(\varphi, \varphi).$$

The process $\xi(\varphi)$ can then, by continuity, be extended to X. It is clear that the maximal extension of K_T is the closure of K_T in the metric $\|\varphi\|^2 = B(\varphi, \varphi)$, which we denote by $L^2_T(B)$. The extension of $\xi(\varphi)$ to $L^2_T(B)$ makes it possible to describe the elements of $H_T(\xi)$ as follows: $H_T(\xi) \ni \eta = \xi(q)$, $q \in L^2_T(B)$.

Definition. Let E_t be a family of projection operators in $H_T(\xi)$ onto the subspaces $H_t(\xi)$, $t \leq T$. Following Hida–Cramér, we shall call the process $\xi(\varphi)$, $\varphi \in K_T$, a process of multiplicity 1 if there exists an element $\eta \in H_T(\xi)$ for which the linear span $\{E_t\eta = \eta_t,\ 0 \leq t \leq T\}$ is dense in $H_T(\xi)$ (cf. ([8])); the element η will be called cyclic.

Theorem 1. Let $\xi(\varphi)$, $\varphi \in K_T$, be a Gaussian process of multiplicity 1; $\eta \in H_T(\xi)$ a cyclic element, $\eta_t = E_t\eta = \xi(q_t)$, $q_t \in L^2_T(B) \cap$

$\eta L^2_t(B)$ and $f_t = U\eta_t \in H_t(B)$, $f_\varphi(\cdot) = B(\varphi, \cdot) \in H_T(B)$. Then $\xi(\varphi)$ admits the canonical representation

$$\xi(\varphi) = \int_0^\tau \frac{df_t(\varphi)}{d\sigma(t)}\, d\eta_t = \int_0^\tau \frac{df_\varphi(q_t)}{d\sigma(t)}\, d\eta_t, \quad \varphi \in K_\tau, \tag{1}$$

where the integral is defined as a stochastic integral with respect to the process η_t with independent increments, $\sigma(t) = M|\eta_t|^2 = B(q_t, q_t) \geq 0$ is a nondecreasing bounded function, and $df(\cdot)/d\sigma(\cdot)$ is understood as the derivative in the Radon-Nikodym sense.

Theorem 2. Let $B(\varphi, \psi)$ be the correlation functional of a generalized Gaussian process $\xi(\varphi)$, $\varphi \in K_T$, continuous in the extension $X \supset K_T$. If for some $f_0 \in X'$ and every t, $0 \leq t \leq T$, there exists an element $q_t \in X_t$, where X_t is the closure of K_t in X, such that

$$B(\varphi, q_t) = f_0(\varphi), \quad \varphi \in K_t, \tag{2}$$

then $f_0(q_t) = \sigma(t) \geq 0$ defines a nondecreasing bounded function of t, and the functions $\hat{f}(t) = f(q_t)$ for $f \in H_T(B)$ are absolutely continuous with respect to $\sigma(t)$, moreover $\hat{f}(t)$ and $\sigma(t)$ do not depend on the nonunique choice of q_t. If the linear span of $\{q_t, 0 \leq t \leq T\}$ is dense in X, then $H_T(B)$ is unitarily isomorphic to $L^2([0,T], d\sigma(t))$, and

$$\langle f_1, f_2 \rangle_{H_T(B)} = \int_0^T \frac{d\hat{f}_1(t)\, d\hat{f}_2(t)}{d\sigma(t)}. \tag{3}$$

3. Consider a generalized stationary Gaussian process $\xi(\varphi)$, $\varphi \in K_T$, with mean value $M\xi(\varphi) = 0$ and correlation functional $B_\mu(\varphi, \psi) = \int \widetilde{\varphi}(\lambda)\overline{\widetilde{\psi}(\lambda)}|\lambda|^{-\mu} d\lambda$, $\mu < 1$, where $\widetilde{\varphi}(\lambda)$ is the Fourier transform of the function $\varphi(t)$.

Theorem 3. 1) The integral equation

$$\int \widetilde{\varphi}(\lambda)\, \overline{\widetilde{q}_t(\lambda)} |\lambda|^{-\mu} d\lambda = \int_0^t \varphi(t)\, dt, \quad \varphi \in K_t, \tag{4}$$

has the solution

$$q_t(s) = \frac{1}{2\pi} \frac{1}{\Gamma(1-\mu)} (t-s)_+^{-\mu/2} s_+^{-\mu/2}, \quad 0 < s \leq T, \tag{5}$$

($u_+ = u$ for $u \geq 0$ and $u_+ = 0$ for $u \leq 0$); the family $\{q_t(s), 0 \leq t \leq T\}$ is dense in $L^2(0,T)$.

2) For $0 \leq \mu < 1$, the elements of $H_T(B_\mu)$ are $f(\varphi) = \int_0^T f(t)\varphi(t)\, dt$, $f(t) \in L^{2/(1-\mu)}$, and

$$\langle f_1, f_1 \rangle_{H_T(B_\mu)} = \frac{1}{2\pi} \int_0^T \left| t^{\mu/2} D^{\mu/2} [f(t) t^{-\mu/2}] \right|^2 dt, \tag{6}$$

where

$$D^\alpha[f] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s)\, ds$$

is the fractional differentiation operator of order α.

3) For $-1 \leq \mu < 0$, the elements of $H_T(B_\mu)$ are singular generalized functions of first order: $H_T(B_\mu) \ni f(\varphi) = \int_0^T F(t)\varphi'(t)\, dt$, $F(t) \in \mathrm{Lip}((1+\mu)/2)$, if $-1 < \mu < 0$, and $F(t) \in L^p$, $p > 1$, if $\mu = -1$, $\left(\frac{d}{dt}F(t) = f(t)\right)$; in this case

$$\langle f, f\rangle_{H_T(B_\mu)} = \frac{1}{2\pi}\int_0^T \left| t^{\mu/2}\frac{d}{dt}\int_0^t F(s)\, d\left[\frac{(t-s)^{-\mu/2}s^{-\mu/2}}{\Gamma(1-\mu)}\right]\right|^2 dt. \qquad (7)$$

Using Theorem 3 and the method developed in (2), it is easy to write out the Radon-Nikodym derivative of the measures P_1 and P_2 corresponding to the processes $\xi(\varphi)$ and $\xi(\varphi)+m(\varphi)$, $m(\varphi)\in H_T(B_\mu)$.

The process $\xi(\varphi)$ is the derivative of order $([(1+\mu)/2]+1)$ of the Gaussian process $x(t)$ with homogeneous increments, invariant with respect to the similarity transformation (8). This observation makes it possible to transfer Theorem 3 also to the process $x(t)$. Let $|\mu|<1$, and let $x(t)$ be a Gaussian process with correlation function

$$R_\mu(s,t) = k\left(|s|^{1+\mu}+|t|^{1+\mu}-|t-s|^{1+\mu}\right), \qquad k = 2\sin\frac{\mu}{2}\pi\Gamma(-1-\mu).$$

Since $x(t) = \xi(\chi_{0t})$, $H_T(R_\mu)$ consists of functions $F(t) = f(\chi_{0t})$, $f(\varphi)\in H_T(B_\mu)$, with scalar product

$$\langle F_1(t), F_2(t)\rangle_{H_T(R_\mu)} = \langle f_1, f_2\rangle_{H_T(B_\mu)}.$$

Combining Theorem 1 with Theorem 3, we obtain the canonical representation of the process $x(t)$ in the form

$$x(t) = \int_0^t Q(t,s)\, d\eta(s),$$

where

$$Q(t,s) =$$

$$= \frac{1}{\pi}\sin\frac{\mu\pi}{2}\begin{cases}\displaystyle\int_s^t (\tau-s)^{\mu/2-1}\tau^{\mu/2}\, d\tau, & \text{for } 0<\mu<1,\\ \displaystyle\int_s^t (\tau-s)^{\mu/2-1}\left[\tau^{\mu/2}-s^{\mu/2}\right] d\tau + \frac{2}{\mu}(ts-s^2)^{\mu/2}, & \text{for } -1\le\mu<0,\end{cases}$$

and the process with independent increments $\eta(s)$, for which $H_t(x) = H_t(\eta)$, $t\le T$, has the form

$$\eta(s) = \begin{cases}\displaystyle\int_0^s (s-t)^{-\mu/2}t^{-\mu/2}\, dx(t), & \text{for } 0<\mu<1,\\ \displaystyle\int_0^s x(t)\, d\left[(s-t)^{-\mu/2}t^{-\mu/2}\right], & \text{for } -1\le\mu\le 0.\end{cases}$$

Hence, in the usual way, one obtains formulas for prediction of $x(t)$, first obtained in (9).

4. Let $\xi_i(\varphi)$, $\varphi \in K_T$, be Gaussian stationary processes with means $m_i(\varphi) = 0$, correlation functions

$$B_i(\varphi, \varphi) = \psi_i(\varphi * \varphi^*), \qquad \psi_i \in K',$$

$\varphi^*(x) = \varphi(-x)$, and spectral measures $F_i(d\lambda)$ (1); P_i are the measures induced by the processes $\xi_i(\varphi)$, $\varphi \in K_T$, in the space K'_T, $i = 1, 2$.

Theorem 4. If $F_i(d\lambda)$ are absolutely continuous and

$$0 < c_1 < \frac{F_1(d\lambda)}{d\lambda}(1 + |\lambda|)^{2n+\mu} < c_2 \quad \text{as } \lambda \to \infty,$$

where n is an integer and $-1 \le \mu < 1$, then the measures P_i are equivalent if and only if

$$\Delta^{(2n)}(\varphi * \psi^*) \in H_T(B_\mu) \otimes H_T(B_\mu),$$

where

$$\Delta(\varphi) = \psi_1(\varphi) - \psi_2(\varphi), \qquad \Delta^{(k)}(\varphi) = \Delta(\varphi^{(k)}),$$

in other words:

1) for $0 \le \mu < 1$,

$$\Delta^{(2n)}(\varphi * \psi^*) = \int_0^T \int_0^T \Delta^{(2n)}(s - t)\varphi(s)\psi(t)\, ds\, dt,$$

where

$$\Delta^{(2n)}(s - t) \in L^{2/(1-\mu)}([0, T] \times [0, T]),$$

and

$$\int_0^T \int_0^T (st)^\mu \left| D_t^{\mu/2} D_s^{\mu/2} \left[\Delta^{(2n)}(s - t)(st)^{-\mu/2}\right]\right|^2 ds\, dt < \infty;$$

2) for $-1 \le \mu < 0$, the functional

$$\Delta^{(2n-2)}(\varphi) = \int_0^T \Delta^{(2n-2)}(t)\varphi(t)\, dt,$$

$$\Delta^{(2n-2)}(t) \in C[0, T]$$

and

$$\int_0^T \int_0^T (st)^\mu \left| \frac{\partial^2}{\partial s\, \partial t} \int_0^s \int_0^t \Delta^{(2n-2)}(u - v)\, du\, dv\, [(s - u)(t - v)uv]^{-\mu/2} \right|^2 ds\, dt < \infty.$$

The proof is based on the results of ([2,10]).

Theorem 5. Let $f(\lambda) \geq 0$ be a locally summable function and $c_1 < f(\lambda)|\lambda|^{(2n+\mu)} < c$ as $\lambda \to \infty$, $-1 \leq \mu < 1$, n an integer. For the existence of a solution of the Wiener-Hopf integral equation

$$a(t) = \int e^{i\lambda t} \varphi(\lambda) f(\lambda)\, d\lambda, \qquad 0 \leq t \leq T,$$

in the class of functions $L_T^2(f)$—the closure of $\{\varphi(\lambda),\ \widetilde{\varphi}(t) \in K_T\}$, in the metric

$$\int |\varphi|^2 f\, d\lambda = \|\varphi\|^2_{L_T^2(f)},$$

it is necessary and sufficient that $a(t)$ have $n-1$ derivatives and that $a^n(t) \in H_T(B_\mu)$; the n-th derivative is understood as a generalized one.

Theorems 4 and 5 in the case $\mu = 0$ were obtained by Yu. A. Rozanov ([11,12]).

5. Let $\xi(\varphi)$, $\varphi \in K(E^n)$, be a Gaussian homogeneous field with correlation functional

$$B(\varphi, \varphi) = \int_{E^n} |\widetilde{\varphi}(\lambda)|^2 f(\lambda)\, d^n\lambda,$$

and let Ω be a bounded domain with smooth boundary in E^n. The notation $H_\Omega(B)$ has the same meaning as in the one-dimensional case.

Theorem 6. Let $0 < c_1 < f(\lambda)(1 + |\lambda|^2)^{-\mu} < c_2$, $\mu > 0$; then the space $H_\Omega(B)$, up to equivalence of norms, coincides with the Sobolev-Slobodetskii space $W_2^\mu(\Omega)$, whose norm has the form:

$$\|f\|^2_{W_2^\mu(\Omega)} = \|f\|^2_{L^2} + \sum_{|\alpha|=[\mu]} \|f^{(\alpha)}\|^2_{L^2} + \sum_{|\alpha|=[\mu]} \int_{\Omega\times\Omega} \frac{|f^{(\alpha)}(x) - f^{(\alpha)}(y)|^2}{|x-y|^{n+2\beta}}\, d^n x\, d^n y,$$

where

$$\beta = \mu - [\mu], \qquad \alpha = (\alpha_1, \dots, \alpha_n), \qquad |\alpha| = \sum_{i=1}^n \alpha_i$$

and

$$f^\alpha(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f(x)$$

is a generalized derivative in the sense of Sobolev ([13]).

Theorem 6 follows from results of V. M. Babich and L. N. Slobodetskii (see ([13])).

Remark 1. Theorems 4 and 5, taking Theorem 6 into account, admit obvious reformulations for the case of homogeneous fields, one of which has the spectrum indicated above.

Remark 2. Since

$$W_2^\mu(\Omega) \otimes W_2^\mu(\Omega) \supset W_2^{2\mu}(\Omega \times \Omega),$$

from Theorem 4 we obtain sufficient conditions for the equivalence of two homogeneous fields, namely,

$$\Delta(s-t) \in W_2^{2\mu}(\Omega \times \Omega).$$

Remark 3. Comparing Theorems 3, 4, and 6 in the case $E^n = E^1$ makes it possible to describe the space $W_2^{\mu}([0,T])$, $\mu > 0$, in a new way.

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Note: Figure translations are in progress. See original paper for figures.