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Abstract

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ON THE THEORY OF PSEUDODIFFERENTIAL OPERATORS ON A MANIFOLD WITH BOUNDARY

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A theorem is proved on the equivalence of the ellipticity and Fredholm properties for a certain class of pseudodifferential problems in spaces $\mathcal{H}^{s,p}$ of Bessel potentials, which are sections of Hermitian vector bundles over a smooth compact Riemannian manifold with boundary. The problems considered are analogous to the pseudodifferential problems of M. I. Vishik and G. I. Eskin (^{1}) in spaces of vector functions on a smooth compact Euclidean domain. In (^{1}) a broad spectrum of such spaces is indicated in which the ellipticity of a problem implies its Fredholm property. However, Bessel potentials are contained in this spectrum only for $p = 2$.

1. Let M be a compact smooth Riemannian manifold with boundary. (By smoothness we everywhere mean belonging to the class $\mathcal{C}^\infty$.) Let M' be its boundary, $M_I = M \setminus M'$ its interior, $\pi : T \to M$ the tangent bundle of the manifold M, and $\pi_0 : T_0 \to M$ the subbundle of nonzero vectors in T.

By E we denote a Hermitian smooth vector bundle over M. Let $\mathcal{C}^\infty(M_I; E)$ be the space of smooth sections of this bundle over M_I, and $\mathcal{C}_0^\infty(M_I; E)$ the subspace of sections with compact supports in M_I. For $1 < p < \infty$ there is defined the Banach space $\mathcal{L}^p(M; E)$ of distributions—sections of the bundle—with norm

$$\|u\|_p = \left(\int |u|^p dM \right)^{1/p},$$

where dM is the Riemannian volume element on M. With the aid of the Laplace operator on M with coefficients in E, the space $\mathcal{L}^p(M; E)$ generates an interpolation scale of Banach spaces $\mathcal{H}^{s,p}(M; E)$, $-\infty < s < \infty$, of distributions—sections of the bundle E (see the survey (^{3})). We note that $\mathcal{H}^{0,p}(M; E) = \mathcal{L}^p(M; E)$, and for natural m, $\mathcal{H}^{m,p}(M; E)$ is the Sobolev space $\mathcal{W}^{m,p}(M; E)$ of S. L. Sobolev. The closure of $\mathcal{C}_0^\infty(M_I; E)$ in $\mathcal{H}^{s,p}(M; E)$ is denoted by $\mathcal{H}_0^{s,p}(M; E)$.

Let E_1, E_2 be Hermitian smooth bundles over M. The manifold M_I is open. Therefore, for $-\infty < r < \infty$, there is defined a class $\mathfrak{P}^r(M_I; E_1, E_2)$ of pseudodifferential operators P of order r (^{4}). By definition, these operators act from $\mathcal{C}_0^\infty(M_I; E_1)$ to $\mathcal{C}^\infty(M_I; E_2)$. Their symbols are smooth homomorphisms $\sigma(P) : \pi_0^*(E_1|M_I) \to \pi_0^*(E_2|M_I)$. Here and below the vertical bar is the sign of restriction. We single out in $\mathfrak{P}^r(M_I; E_1, E_2)$ the subclass $\mathfrak{P}^r(M; E_1, E_2)$ of operators whose symbols $\sigma(P)$ are restrictions of smooth mappings $\bar{\sigma}(P) : \pi_0^*(E_1) \to \pi_0^*(E_2)$.

Fix an arbitrary number p strictly between 1 and ∞.

Lemma 1. If $P \in \mathfrak{P}^r(M; E_1, E_2)$, then the operator $P : \mathcal{C}_0^\infty(M_I; E_1, E_2) \to \mathcal{C}^\infty(M_I; E_1, E_2)$ (necessarily uniquely) extends to a continuous linear mapping

$$P_{(0)}^s : \mathcal{H}_0^{s,p}(M; E_1) \to \mathcal{H}^{s,p}(M; E_2).$$

Let N be the normal bundle over the boundary M', oriented into M. Denote by N^+ the subbundle of inward normals in N.

Let T' be the tangent bundle of the manifold M', T_0' the subbundle of nonzero elements in T', and $\rho : T_0' \to M'$ the corresponding projection. Setting $E' = E|M'$, define over T_0' the direct product of bundles

$$E^+ = \rho^* N^+ \times \rho^* E',$$

whose space is naturally fibered over $\rho^* N^+$. With this Hermitian bundle $E^+ \to \rho^* N^+$ are associated the Banach bundles of sections of the classes $\mathcal{C}_0^\infty, \mathcal{C}^\infty, \mathcal{H}_0^{s,p}, \mathcal{H}^{s,p}$.

Write the restriction $\bar{\sigma}(P)$ to $T_0|M'$ in the form

$$\bar{\sigma}(P)(\xi) = \bar{\sigma}(P)(\nu, \tau),$$

where (ν, τ) are the normal and tangent components of the vector $\xi \in T_0|M$. This restriction can be regarded as a family of symbols

$$\bar{\sigma}(P)(\cdot, \tau) : (\rho^* N^+)_\tau \to \operatorname{Hom}(E_{1\tau}^+, E_{2\tau}^+)$$

of Wiener–Hopf operators

$$\iota(P)_\tau : \mathcal{C}_0^\infty(\rho^* N_\tau^+; E_{1\tau}) \to \mathcal{C}^\infty(\rho^* N_\tau^+; E_{2\tau}^+), \qquad \tau \in T_0'.$$

The totality of the latter defines a homomorphism of Banach bundles

$$\iota(P) : \mathcal{C}_0^\infty(\rho^* N^+; E_1^+) \to \mathcal{C}^\infty(\rho^* N^+; E_2^+),$$

which we shall call the **indicator** of the operator P.

Lemma 2. For every real s, the indicator of the operator

$$P \in \mathfrak{P}^r(M; E_1, E_2)$$

extends to a homomorphism of Banach bundles

$$\iota_0^s(P) : \mathcal{H}_0^{s,p}(\rho^* N^+; E_1^+) \to \mathcal{H}^{s,p}(\rho^* N^+; E_2^+).$$

2. The restriction operator to the boundary

   $$\delta : \mathcal{C}^\infty(M; E) \to \mathcal{C}^\infty(M'; E')$$

   for $ps > 1$ extends to a continuous **epimorphism of restriction**

   $$\delta : \mathcal{H}^{s,p}(M; E) \to \mathcal{B}^{s-1/p,p}(M'; E'),$$

   where $\mathcal{B}^{s,p}(M'; E')$, $-\infty < s < \infty$, is the scale of O. V. Besov spaces of sections–distributions of the bundle $E' = E|M'$ over M' (see, for example, the survey $(^3)$).

The adjoint **monomorphism of corestriction** (if one replaces $-s$ by s, $1-1/p$ by $1/p$)

$$\delta^* : \mathcal{B}^{s+1-1/p,p}(M'; E') \to \mathcal{H}_0^{s,p}(M; E)$$

is defined for $ps < 1 - p$.

Let

$$P \in \mathfrak{P}^r(M; E_1, E_2).$$

Introduce also an operator

$$P_2 \in \mathfrak{P}^{r_2}(M; E_1|U, F_2),$$

where U is a tubular neighborhood of the boundary M', F_2 is a Hermitian bundle over U, and an operator

$$P_1 \in \mathfrak{P}^{r_1}(M; F_1, E_2|U),$$

where F_1 is a Hermitian bundle over U. Let

$$\varphi \in \mathcal{C}_0^\infty(U), \qquad 0 \le \varphi \le 1, \qquad \varphi|M' = 1.$$

For every real s for which

$$s_1 = s - r + r_1 < -1 + 1/p, \qquad s_2 = s - r_2 > 1/p,$$

define the **fitting** $\mathfrak{P}^s_{(0)}$ of the operator P as the operator

$$\mathfrak{P}_0^{(s)} : \mathcal{H}_0^{s,p}(M; E_1) \times \mathcal{B}^{s_1+1-1/p,p}(M'; F_1') \to \mathcal{H}^{s-r,p}(M; E_2) \times \mathcal{B}^{s_2-1/p,p}(M'; F_2')$$

of the form

$$\mathfrak{P}^s_{(0)}(u, v) = (P^s_{(0)}u + \varphi P^{s_1}_{1(0)}\delta^* v,\ \delta P^{s_2}_{2(0)}(\varphi u));$$

here

$$F_1' = F_1|M', \qquad F_2' = F_2|M'$$

(cf. (1), and also (2)).

Similarly one defines the **fitting of the indicator** $\iota(P)$

$$\iota^s_{(0)}(P) : \mathcal{H}_0^{s,p}(\rho^* N^+; E_1^+) \times F_1' \to \mathcal{H}^{s-r,p}(\rho^* N^+; E_2^+) \times F_2',$$

which is a homomorphism of the form

$$\iota^s_{(0)}(P)(u,v) = (\iota^s_{(0)}(P)u + \iota^{s_1}_{(0)}(P_1)\delta^* v,\ \delta\, \iota^s_{(0)}(P)u);$$

here δ, δ^* are the corresponding homomorphisms of restriction and corestriction on the bundle N^+.

The fitting $\mathfrak{P}^s_{(0)}$ is called **elliptic** if the homomorphisms $\sigma(P)$ and $\iota^s_{(0)}(P)$ are isomorphisms.

A linear operator is called **Fredholm** if its range is closed and the codimension of this range and the dimension of the null-space of the operator are finite.

Theorem 1. The fitting $\mathfrak{P}^s_{(0)}$ is a Fredholm operator if and only if it is elliptic.

Corollary 1. The operator $P^s_{(0)}$ is Fredholm if and only if the homomorphisms $\sigma(P)$ and $\iota^s_{(0)}(P)$ are isomorphisms.

3. If $P \in \mathscr{P}^r(M; E_1, E_2)$, then its adjoint P^* belongs to $\mathscr{P}^r(M; E_2, E_1)$. In this item we consider operators P satisfying the following condition:

($\mathscr{D}$) The operator P^* maps $\mathscr{C}^\infty_0(M; E_2)$ into $\mathscr{C}^\infty_0(M; E_1)$.

This condition is satisfied, for example, by differential operators.

Under this assumption the operator $(P^*)^{-s+r}_{(0)}$ acts from

$$\mathscr{H}^{-s+r,p}_0(M; E_2)$$

to $\mathscr{H}^{-s,p}_0(M; E_1)$. Therefore its adjoint is defined,

$$P^s : \mathscr{H}^{s,p}(M; E_1) \to \mathscr{H}^{s-r,p}(M; E_2).$$

Lemma 3. The operator P satisfies condition ($\mathscr{D}$) if and only if the indicator $\iota(P^*)$ maps $\mathscr{C}^\infty_0(\rho^* N^+; E_2^+)$ into $\mathscr{C}^\infty_0(\rho^* N^+; E_1^+)$.

As above, the homomorphism dual to such an indicator defines, for every real s, a mapping

$$\iota^s(P) : \mathscr{H}^{s,p}(\rho^* N^+; E_1^+) \to \mathscr{H}^{s-r,p}(\rho^* N^+; E_2^+).$$

Let now P_1, P_2 be the same as in the preceding item, but let P satisfy condition ($\mathscr{D}$). Then, analogously to item 2, with the aid of P^s, $P^{s_1}_{1(0)}$, $P^{s_2}_2$ one can define the ($\mathscr{D}$)-equipment of the operator P,

$$\mathfrak{P}^s : \mathscr{H}^{s,p}(M; E_1) \times \mathscr{B}^{s_1+1-1/p}(M'; F_1') \to \mathscr{H}^{s-r,p}(M; E_2) \times \mathscr{B}^{s_2-r-1/p,p}(M'; F_2'),$$

and, with the aid of $\iota^s(P)$, $\iota^{s_1}_{(0)}(P_1)$, $\iota^{s_2}(P_2)$, the ($\mathscr{D}$)-equipment of the indicator $\iota(P)$,

$$\iota^s(P) : \mathscr{H}^{s,p}(\rho^* N^+; E_1^+) \times F_1' \to \mathscr{H}^{s-r,p}(\rho^* N^+; E_2^+) \times F_2'.$$

We shall call the equipment P^s elliptic if the homomorphisms $\sigma(P)$ and $\iota^s(P)$ are isomorphisms.

Theorem 2. The equipment $\mathfrak{P}^s$ is a Fredholm operator if and only if it is elliptic.

Corollary 2. The operator P^s is Fredholm if and only if the homomorphisms $\sigma(P)$ and $\iota^s(P)$ are isomorphisms.

4. M. I. Vishik and G. I. Eskin (^{1}) established under what conditions an operator $P \in Pr(M; E_1, E_2)$ admits an elliptic equipment $P^s_{(0)}$, when M is a Euclidean domain. A direct generalization of their result is the following

Proposition 1. In order that an operator $P \in \mathcal{P}^r(M; E_1, E_2)$ admit a Fredholm equipment $P^s_{(0)}$, it is necessary and sufficient that its symbol $\sigma(P)$ be an isomorphism, that the indicator operators $\iota^s(P)_\tau$, $\tau \in T_0'$, be Fredholm, and that the index of the homomorphism $\iota^s(P)$ in the sense of (^{5}) belong to the inverse image $\rho^* K(M')$ in $K(T_0')$ with respect to the canonical projection $\rho : T_0' \to M$.

One can compute the index of the Wiener–Hopf family of operators. As a consequence one obtains the following simple sufficient conditions:

Proposition 2. An operator $P \in Pr(M; E_1, E_2)$ admits a Fredholm equipment if the following set of conditions is satisfied: (a) M is a domain in R^n; (b) $\dim E_1 = \dim E_2 < (n-1)/2$; (c) the symbol $\sigma(P)$ is an isomorphism; (d) for every $\nu \in N$ the spectrum of the endomorphism

$$[\bar{\sigma}(P)(\nu, 0)]^{-1}[\bar{\sigma}(P)(-\nu, 0)] \in \operatorname{End} E_{1\pi(\nu)}$$

does not contain numbers with argument $2\pi/p$.

Remark added in proof. If n is odd, condition (b) is superfluous.

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