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MATHEMATICS

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Abstract

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MATHEMATICS

L. S. FRANK

ON HYPERBOLIC DIFFERENCE OPERATORS

(Presented by Academician G. I. Petrov, 24 IX 1968)

1. The class of strictly hyperbolic differential operators, first introduced and studied in (^{1}) and later in (^{2}, ^{3}), is characterized by several remarkable properties. First of all, the Cauchy problem for such operators is well posed both when time is varied forward and backward. In addition, the solutions of the direct and inverse Cauchy problems for hyperbolic equations satisfy energy inequalities which, in certain special cases (equations of mathematical physics describing oscillatory processes, symmetric and symmetrizable first-order hyperbolic systems), turn into equalities having the meaning of conservation laws. It is natural to require of difference operators approximating hyperbolic differential equations that they preserve the basic qualities inherent in the latter. However, most difference approximations used for the numerical solution of hyperbolic equations possess so-called scheme viscosity and, as a rule, do not satisfy the above requirements.

There arises a need to single out a class of difference operators that preserve, on a sequence of difference grids, the basic properties of hyperbolic differential operators. The present note is devoted precisely to this question.

2. Function spaces. In the R^{n+1} variables $(t, x) = (t, x_1, \dots, x_n)$ we consider a one-parameter sequence of difference grids

$$R_h^{n+1} = \{(t, x) : t = p_0 r_0 h,\ x_j = p_{jr_{jh}}\},$$

where p_j are integers, $-\infty < p_j < \infty$, and r_j are positive constants, $0 \le j \le n$. In all that follows we assume that $0 < h \le h_0$, where h_0 is sufficiently small. Denote by $H_{s,l}^{[t_0,t_1]}$ (s and l are nonnegative integers) the set of all grid functions $u_h(t, x)$ for which

$$D_t^k u_h(t, x) \in H_s^*(R_h^n)$$

for $0 \le k \le l,\ t_0 \le t \le t_1 - k r_0 h$, where

$$D_t u_h(t, x) = \frac{u_h(t + \tau, x) - u_h(t, x)}{i\tau}.$$

(For the definition of the space $H_s^*(R_h^n)$, see (^{5}).)

3. Definition of strictly hyperbolic difference operators. We first define homogeneous strictly hyperbolic difference operators. Let $h \in R^1_+$, $\lambda \in C^1$, $\xi \in C^n$. Put

$$V^n = \{(h,\lambda,\xi) : |1 + ihr_0\lambda| = 1,\ |1 + ihr_j\xi_j| = 1,\ 1 \leq j \leq n\}. \tag{1}$$

Denote by $P_m(t,x;h,\lambda,\xi,\bar{\xi})$, $m \geq 1$, a function defined on $R^{n+1} \times V^n$ and possessing the following properties:

1°. P_m is a polynomial in λ of degree m.

2°. P_m is a homogeneous function of order m on the manifold V^n; in other words, for any $\alpha \in R^1_+$

$$P_m(t,x;\alpha^{-1}h,\alpha\lambda,\alpha\xi,\alpha\bar{\xi}) = \alpha^m P_m(t,x;h,\lambda,\xi,\bar{\xi}). \tag{2}$$

3°. P_m is infinitely differentiable with respect to the variables $(\xi,\bar{\xi})$ for $\xi \neq 0$, and with respect to the variables (t,x) is uniformly bounded in h, together with all its

difference derivatives on the grid R^{n+1}_h; moreover, $P_m(t,x;h,\lambda,\xi,\bar{\xi}) \equiv P_m(\infty;h,\lambda,\xi,\bar{\xi})$ for $|t|^2 + |x|^2 \geq R^2$, where R is a sufficiently large constant.

4°. Finally, the roots $\mu^{(j)}(t,x;\omega,\bar{\omega})$, $1 \leq j \leq m$, of the equation

$$P_m(t,x;1,\mu,\omega,\bar{\omega}) = 0 \tag{3}$$

for $|1 + i\omega_k| = 1,\ 1 \leq k \leq n$, are all distinct and satisfy the conditions

$$\left|1 + i\mu^{(j)}(t,x;\omega,\bar{\omega})\right| = 1 \quad \text{for } |1 + i\omega_k| = 1, \tag{4}$$

$$1 \leq k \leq n, \quad 1 \leq j \leq m;$$

$$\left|\mu^{(j)}(t,x;\omega,\bar{\omega}) - \mu^{(k)}(t,x;\omega,\bar{\omega})\right| \geq \delta > 0 \quad \text{for } j \neq k, \tag{5}$$

where δ does not depend on t, x, and ω. Relations (4), (5) are analogous to the hyperbolicity condition of I. G. Petrovsky for differential equations.

The functions P_m satisfying conditions 1°–4° are called homogeneous strictly hyperbolic functions of order m. They define on grid functions $u_h(t,x) \in H^{[t_0,t_1]}_{s,l}$, $s \geq m$, $l \geq m$, strictly hyperbolic difference operators in convolutions by the formula

$$P_m u_h(t,x) = F^{-1}_{\xi\to x} P_m(t,x;h,D_t,\xi,\bar{\xi}) F_{x\to\xi} u_h(t,x), \tag{6}$$

where $\xi = (\xi_1, \dots, \xi_n)$, $\xi_k = (e^{ihr_k\xi_k} - 1)/ir_{kh}$, and $F_{x\to\xi}$ and $F^{-1}_{\xi\to x}$ are respectively the direct and inverse Fourier transforms on the grid R^n_h of the variables $x = (x_1, \dots, x_n)$; the operator D_t was defined above.

We shall also consider nonhomogeneous functions $P(t, x; h, \lambda, \xi, \bar{\xi})$, representable in the form

$$P(t, x; h, \lambda, \xi, \bar{\xi}) = \sum_{k=1}^{m} P_k(t, x; h, \lambda, \xi, \bar{\xi}) + P_0(t, x; h, \xi, \bar{\xi}), \tag{7}$$

where P_k, $1 \le k \le m$, are homogeneous functions satisfying conditions 1°–3°, and P_m satisfies, in addition, the hyperbolicity condition 4°; for the function P_0 (generally speaking, nonhomogeneous in the variables $h, \xi, \bar{\xi}$) it is assumed that condition 3° holds and that it is uniformly bounded with respect to h, ξ, t, x when $\xi_k = (e^{ihr_k\xi_k} - 1)/ihr_k$. The functions P_k, $0 \le k \le m$, define by formula (6) the corresponding operators; here the operator P_0 (corresponding to the function P_0) is bounded in the spaces $H^*_s(R^n_h)$ for all s. We note that, for difference schemes approximating hyperbolic differential equations, the functions P_k are, as a rule, polynomials not only in λ, but also in $(\xi, \bar{\xi})$.

4. The Cauchy problem for strictly hyperbolic difference operators. The energy inequality

The direct Cauchy problem for hyperbolic difference operators, just as in the differential case, consists in finding a function $u^+_h(t, x)$ satisfying, for $0 \le t \le T_1$, the equation

$$P u^+_h(t, x) = f^+(t, x), \quad 0 \le t \le T_1, \tag{8}$$

and the initial conditions

$$D^k_t u^+_h(t, x)\big|_{t=0} = \varphi^+_k(x), \quad 0 \le k \le m - 1. \tag{9}$$

The inverse Cauchy problem is posed for negative values of t and consists in finding a function $u^-_h(t, x)$ satisfying, for $-T_2 \le t \le 0$, the equation

$$P(1 - i\tau\bar{D}_t)^m u^-_h(t, x) = f^-(t, x), \quad -T_2 \le t \le 0, \tag{10}$$

and the initial conditions

$$\bar{D}^k_t u^-_h(t, x)\big|_{t=0} = \varphi^-_k(x), \quad 0 \le k \le m - 1, \tag{11}$$

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where the operator $\bar{D}_t$ is formally adjoint to D_t and is defined by the equality

$$\bar{D}_t u_h(t,x) = (u_h(t,x) - u_h(t-\tau,x))/i\tau.$$

Theorem 1. For solutions of problem (8), (9), the following energy inequality holds

$$\sum_{j=0}^{s+m-1} \left\| D_t^j u_h^+(t) \right\|_{s+m-1-j} \le C \left[\sum_{j=0}^{m-1} \left\| \varphi_j^+ \right\|_{s+m-1-j} + \sum_{j=0}^{s-1} \left\| D_t^j f^+(0) \right\|_{s-1-j} + \sum_{0<t'\le t} \sum_{j=0}^{s} \left\| D_t^j f^+(t') \right\|_{s-j} \tau_0 h \right] \tag{12}$$

where $0 \le t \le T_1$, $\| \cdot \|_s$ is the norm in the space $H_s^*(R_h^n)$, $s \ge 0$, s is an integer, and C does not depend on h or u_h^+. An analogous estimate holds for solutions of problem (10), (11).

In the proof of Theorem 1, a certain modification of the idea of Leray—Gårding is used, which proved very useful for obtaining a similar inequality in the differential case. It consists in the use of the so-called separating polynomial for obtaining energy estimates (see $(^2)$).

Theorem 2. If $T_1 < \infty$, and $f^+(t,x)$ and $\varphi_j^+(x)$ are such that for them the right-hand side of inequality (12) is finite for some integer s, $s \ge 0$, then a solution of problem (8), (9) exists. An analogous assertion can be formulated, for finite T_2, for problem (10), (11).

In the proof of Theorem 2, the energy inequality (12) is used for the adjoint operator P^* (which will also be hyperbolic in the sense of item 4°) and the Hahn—Banach theorem on the extension of a linear continuous functional.

Theorem 3. If for the operator P the estimate (12) holds and the corresponding estimate for solutions of the inverse Cauchy problem holds for some T_1, T_2, and s, $s \ge 0$, then the roots of equation (3) for the principal part P satisfy condition (4).

In proving Theorem 3, solutions of a special form are substituted into the corresponding energy inequalities, indicating the impossibility of such an estimate when condition (4) is not fulfilled. Theorem 3 means that the strict hyperbolicity conditions (4), (5) of the operator P are almost necessary for the uniform well-posedness of the Cauchy problem in both directions and for the fulfillment of the corresponding energy inequalities.

Institute of Space Research Academy of Sciences of the USSR Moscow

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