A NEW ALGORITHM FOR OBTAINING HIGHER APPROXIMATIONS IN THE AVERAGING METHOD

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Abstract

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MATHEMATICS

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A NEW ALGORITHM FOR OBTAINING HIGHER APPROXIMATIONS IN THE AVERAGING METHOD

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Consider a system of ordinary differential equations in the standard form ($^{1-3}$)

$$dx/dt = \varepsilon X(t, x), \tag{1}$$

where $\varepsilon > 0$ is a small parameter; t is time; x, X are elements of the Euclidean space E_n. In the domain $Q\{t \leq 0,\ x \in E_n\}$ the vector-function $X(t, x)$ is defined, Riemann integrable in t, and periodic with period $T = \text{const} \sim 1$; it is continuously differentiable k times with respect to x, and there exist constants M, λ_i such that

$$|X(t, x)| \leq M, \quad |\partial^i X(t, x)/\partial x^i| \leq \lambda_i \quad (i = 1, 2, \dots, k). \tag{2}$$

Averaging the right-hand side of (1) with respect to time, we define the first approximation $\xi(\tau)$ as a function of $\tau = \varepsilon t$ by the equation (^{1})

$$d\xi/dt = \varepsilon X_0(\xi), \quad \xi(0) = x(0). \tag{3}$$

In (3), (5), respectively, the notation used is

$$X_0(\xi) = \frac{1}{T}\int_0^T X(t, \xi)\, dt, \quad \overline{X}(t, \xi) = X(t, \xi) - X_0(\xi). \tag{4}$$

By virtue of (3), (4), we form the expressions *

$$\bar{x} = \xi + \varepsilon \widetilde{X}(t, \xi), \quad \widetilde{X}(t, \xi) = \int_0^t \overline{X}(t, \xi)\, dt. \tag{5}$$

In (4), (5) the integration is with respect to t with $\xi = \text{const}$; therefore $\widetilde{X}$ is periodic in t with period T. Introduce the variable x_1 by the formula

$$x = \bar{x} + \varepsilon x_1. \tag{6}$$

Below we shall show that, in turn, $x_1 = \bar{x}_1 + \varepsilon x_2$, $x_2 = \bar{x}_2 + \varepsilon x_3, \dots,$ $x_{k-1} = \bar{x}_{k-1} + \varepsilon x_k$, and the solution of (1) has the form of the series

$$x = \bar{x} + \varepsilon\{x_1 + \varepsilon[\bar{x}_2 + \dots + \varepsilon(\bar{x}_{k-1} + x_k)]\}. \tag{7}$$

Here, for $k = 1, \dots, p$ in (2), $x_1, \dots, x_p$, respectively, will satisfy equations of the form (1); therefore, for each x_{p-1}, analogously to (5), one can define $\bar{x}_{p-1}$ as a bounded function ** of τ, t, ε, periodic in t with period T, and a function $x_p(t)$, bounded for $0 \leq t \leq O(1/\varepsilon)$.

* Here $\bar{x}$ differs from the improved first approximation (^{1}) in that in the integral (5) the lower limit is not an arbitrary constant but zero, which is convenient for what follows.

** The expression in braces in (7) differs from the higher approximations ($^{1-3}$) by the simplicity of its structure. The principal possibility of a difference is that the coefficients in the expansion of x in powers of ε depend on ε, and hence their determination is not unique. In the algorithm (7) the main identity of equations (1), (8), (21), ⋯for $x, x_1, x_2, \dots$, which made it possible to obtain the error estimate (34) of the k-th approximation under a scheme previously proposed by the author (^{4}) for the first approximation of the averaging method, is essential.

By virtue of (1)–(6) we obtain the differential equation for x_1

$$dx_1/dt = \varepsilon X_1(t, x_1, \tau, \varepsilon), \qquad x_1(0) = 0, \tag{8}$$

where

$$X_1(t, x_1, \tau, \varepsilon) = \frac{X(t, x) - X(t, \xi)}{\varepsilon} - \frac{\partial \widetilde{X}(t, \xi)}{\partial \xi} X_0(\xi) \tag{9}$$

is a periodic function of t with period T, bounded as $\varepsilon \to 0$, if ξ, x_1 are bounded and in (2) $k \geq 2$. Indeed, for $k = 1, 2$ in (2), for $X(t, x)$ the one- and two-term Taylor expansions in powers of $x - \xi = \varepsilon(\widetilde{X} + x_1)$ are valid, from which and (9) it follows that

$$X_1(t,x_1,\tau,\varepsilon)+\frac{\partial\widetilde{X}(t,\xi)}{\partial\xi}X_0(\xi)=\begin{cases}\dfrac{\partial X(t,z_1)}{\partial z_1}(\widetilde{X}+x_1),\\ \dfrac{\partial X(t,\xi)}{\partial\xi}(\widetilde{X}+x_1)+\dfrac{\varepsilon}{2}\dfrac{\partial^2 X(t,z_2)}{\partial z_2^2}(\widetilde{X}+x_1)^2,\end{cases} \tag{10}$$

(10')

where

$$z_i=\xi+\theta_i\varepsilon(\widetilde{X}+x_1),\qquad 0<\theta_i<1,\qquad i=1,2, \tag{11}$$

are intermediate values of the variables in the remainder terms of the Taylor series. In view of (2), the first and second derivatives of X with respect to $z=\xi,z_1,z_2$ are bounded by the constants λ_1,λ_2, respectively.

From (2) and the properties of the functions $X,\overline{X},\widetilde{X}$, periodic in t, it follows that

$$\frac{1}{T}|\widetilde{X}(t,\xi)|,\ |X_0(\xi)|\leq M,\qquad |\partial\widetilde{X}(t,\xi)/\partial\xi|\leq\lambda_1 T. \tag{12}$$

Passing from (8) to the equivalent integral equation*, representing the solution of the latter in the form of a Picard successive-approximation series, taking into account (12) and $k=1$ in (2), we obtain the majorant estimates [4], valid for any $\varepsilon>0$,

$$|x_1|\leq 2TM(\exp\varepsilon\lambda_1 t-1),\qquad \varepsilon|\widetilde{X}+x_1|\leq\varepsilon TM(2\exp\varepsilon\lambda_1 t-1), \tag{13}$$

whence it is seen that $|x_1|\sim 1,\ |x-\xi|\sim\varepsilon$ on intervals $t\sim 1/\varepsilon$. The Lipschitz constants for (1), (8) coincide, since by virtue of (6), (9)

$$\partial X_1/\partial x_1=\partial X/\partial x. \tag{14}$$

Let us estimate (10) with the aid of (12) and bound the result from above by a constant M_1:

$$|X_1|\leq\lambda_1|x_1|+2\lambda_1 TM\leq M_1, \tag{15}$$

which holds on an interval $[0,t^*]$, in general finite, on which $|x_1(t)|\sim 1$, i.e. $t^*\sim 1/\varepsilon$.

Substituting $(10')$ with $\varepsilon=0$ into (8) and averaging, we obtain

$$d\xi_1/dt = \varepsilon X_1^0(\xi_1, \tau), \qquad \xi_1(0) = 0, \tag{16}$$

where

$$X_1^0(\xi_1, \tau) = \frac{1}{T}\int_0^T X_1(t, \xi_1, \tau, 0)\,dt = a(\tau)\xi_1 + b(\tau); \tag{17}$$

$$a(\tau) = \frac{1}{T}\int_0^T \frac{\partial X(t, \xi(\tau))}{\partial \xi}\,dt, \qquad b(\tau) = \frac{1}{T}\int_0^T \frac{\partial X(t, \xi(\tau))}{\partial \xi}\,\widetilde{X}(t, \xi(\tau))\,dt.$$

By analogy with (5) we form the expressions

$$\bar{x}_1 = \xi_1 + \varepsilon\widetilde{X}_1(t, \xi_1, \tau, \varepsilon), \qquad \widetilde{X}_1(t, \xi_1, \tau, \varepsilon) = \int_0^t \overline{X}_1(t, \xi_1, \tau, \varepsilon)\,dt, \tag{18}$$

where

$$\overline{X}_1(t, \xi_1, \tau, \varepsilon) = X_1(t, \xi_1, \tau, \varepsilon) - X_1^0(\xi_1, \tau). \tag{19}$$

In (17), (18) the integration is with respect to t, with ξ_1, τ constant; the properties of the functions $\overline{X}_1, \widetilde{X}_1$ are analogous to those of $\overline{X}, \widetilde{X}$.

Introducing the variable x_2 by the formula

$$x_1 = \overline{x}_1 + \varepsilon x_2 \tag{20}$$

and taking into account (8), (16)–(19), we obtain the equation

$$dx_2/dt = \varepsilon X_2(t, x_2, \tau, \varepsilon), \qquad x_2(0) = 0, \tag{21}$$

where

$$X_2(t, x_2, \tau, \varepsilon) = \frac{X_1(t, x_1, \tau, \varepsilon) - X_1(t, \xi_1, \tau, \varepsilon)}{\varepsilon} - f(t, \xi_1, \tau, \varepsilon),$$

$$f(t, \xi_1, \tau, \varepsilon) = \frac{\partial \widetilde{X}_1(t, \xi_1, \tau, \varepsilon)}{\partial \xi_1} X_1^0(\xi_1, \tau) + \frac{\partial \widetilde{X}_1(t, \xi_1, \tau, \varepsilon)}{\partial \xi} X_0(\xi). \tag{22}$$

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* For more detail, see the analogous passage from (21) to (25).

Equation (21) will have the form (8) if the expression (22) is bounded as $\varepsilon \to 0$; for this it is sufficient that $X_1(t, x_1, \tau, \varepsilon)$ be continuously differentiable with respect to x, i.e. $k = 2$ in (2). Substituting into (22) the one- and two-term Taylor expansions of $X_1(t, x_1, \tau, \varepsilon)$ in powers of $(x_1 - \xi_1) = \varepsilon(\widetilde{X}_1 + x_2)$, which correspond to $k = 2, 3$ in (2), we obtain the expressions

$$X_2(t, x_2, \tau, \varepsilon) + f(t, \xi_1, \tau, \varepsilon) = \begin{cases} \dfrac{\partial X_1(t, z_1, \tau, \varepsilon)}{\partial z_1}(\widetilde{X}_1 + x_2), \\ \dfrac{\partial X_1(t, \xi_1, \tau, \varepsilon)}{\partial \xi_1}(\widetilde{X}_1 + x_2) + \dfrac{\varepsilon}{2}\dfrac{\partial^2 X_1(t, z_2, \tau, \varepsilon)}{\partial z_2^2}(\widetilde{X}_1 + x_2)^2, \end{cases} \tag{23, 23'}$$

where, by z_1, z_2, analogously to (11), intermediate values of the variables in the remainders of the Taylor series are denoted. By virtue of (2), (6), (9), (14), the first and second derivatives of X_1 with respect to $z = \xi_1, z_1, z_2$ are bounded by constants λ_1, λ_2, respectively.

From (2), (15) and the properties of the functions $X_1, \overline{X}_1, \widetilde{X}_1$ that are periodic in t, inequalities analogous to (12) follow:

$$\frac{1}{T}\left|\widetilde{X}_1\right|, \ \left|X_1^0\right| \le M_1, \qquad \left|\frac{\partial \widetilde{X}_1}{\partial \xi_1}\right| \le \lambda_1 T. \tag{24}$$

We pass from (21) to the equivalent integral equation

$$x_2 = \varepsilon \int_0^t X_2(t, x_2, \tau, \varepsilon)\, dt \tag{25}$$

and represent the solution of (25) in the form of a series of successive Picard approximations

$$x_2 = x_2^0 + (x_2^1 - x_2^0) + \cdots + (x_2^n - x_2^{n-1}) + \cdots, \tag{26}$$

where

$$x_2^0 = 0, \qquad x_2^n = \varepsilon \int_0^t X_2(t, x_2^{n-1}, \tau, \varepsilon)\, dt. \tag{27}$$

Using (24) and $k = 2$ in (2), we establish the inequalities

$$|x_2^1| \le TM^1\lambda_1\varepsilon t, \dots, \ |x_2^n - x_2^{n-1}| \le TM^1\frac{(\lambda_1\varepsilon t)^n}{n!}, \dots; \qquad M^1 = 2M_1 + M. \tag{28}$$

Hence the series (27) converges uniformly with respect to x_2, and the estimate holds

$$|x_2| \leq TM^1(\exp \lambda_1 \varepsilon t - 1), \tag{29}$$

from which it is seen that on intervals $t \sim 1/\varepsilon$, $x_2 \sim 1$, and

$$|x_1 - \xi_1| = \varepsilon|\widetilde{X}_1 + x_2| \leq \varepsilon T\left[M^1(\exp \lambda_1 \varepsilon t - 1) + M_1\right] \sim \varepsilon. \tag{30}$$

With the aid of (14), (20)–(22) it is easy to show the coincidence of the Lipschitz constants for (1), (21). The estimate for (23), analogously to (15), is fulfilled in ob–

seek on the finite interval $[0, t^*]$, where $t^* \sim 1/\varepsilon$:

$$|X_2| \leq \lambda_1|x_2| + \lambda_1 TM^1 \leq M_2. \tag{31}$$

Consider the functions

$$\overline{S}_1 = \overline{x} + \varepsilon\overline{x}_1, \qquad S_1 = \overline{x} + \varepsilon\xi_1 \qquad (\overline{S}_1 = S_1 + \varepsilon^2\mathfrak{X}_1), \tag{32}$$

for which, by virtue of (5), (6), (12), (18)–(20), (29), (30), (32), on intervals $t \sim 1/\varepsilon$ we have

$$x - \overline{S}_1 = \varepsilon^2 x_2 \sim \varepsilon^2, \qquad x - S_1 = \varepsilon^2(\mathfrak{X}_1 + x_2) \sim \varepsilon^2. \tag{33}$$

Consequently, S_1 is the second approximation and consists of the first approximation $\xi \sim 1$ and terms $\sim \varepsilon$: the vibrational term $\varepsilon\mathfrak{X}$ and the smooth term $\varepsilon\xi_1$.

Expression (23′) and the constant M_2 in (31) are written out for the construction of ξ_2, $\varepsilon\mathfrak{X}_2$, x_3, and of the third approximation S_2, for which it is sufficient to take $k = 3$ in (2). Obviously, for x_3 we obtain an equation of the form (1); the same applies to x_p, $p \leq k$. As a result, the algorithm (7) and the generalizing estimate (4) hold:

Theorem. Under condition (2), and for any $\varepsilon > 0$, for the difference between the solution of (1) and the k-th approximation S_{k-1} the following majorant estimate is valid*

$$|x - S_{k-1}| \leq C\varepsilon^k, \qquad 0 \leq t \leq O(1/\varepsilon), \tag{34}$$

where C is a constant; $\overline{S}_{k-1} = S_{k-1} + \varepsilon^k\mathfrak{X}_{k-1}$ is the part of the series (7) when $x_k = 0$; $\mathfrak{X}_{p-1}$ is defined by a relation of the form (5), (18).

Remark 1. The requirements on the right-hand side of (1) can be weakened: Riemann integrability in t may be replaced by Lebesgue integrability, and in x, instead of k, one may restrict oneself to $(k-1)$ continuous derivatives and fulfillment, for almost every t, of the Lipschitz condition for the derivative of order $(k-1)$,

$$|F_k(t,x') - F_k(t,x'')| \leq \lambda_k |x' - x''|, \qquad F_k(t,x) = \begin{cases} \dfrac{\partial^{k-1} X(t,x)}{\partial x^{k-1}}, & k = 2, \dots, \\ X(t,x), & k = 1. \end{cases} \tag{35}$$

Then estimate (34) will be valid almost everywhere on the interval $0 \leq t \leq O(1/\varepsilon)$. Indeed, in (35) $F_k(t,x)$ is a function of x of bounded variation, Lebesgue measurable in t, and $\partial F_k/\partial x$ exists almost everywhere in the domain Q and is bounded by the constant λ_k. Consequently, almost everywhere in the domain Q, the variables $x_1, \dots, x_k$ will satisfy differential equations (8), (21), etc., whose right-hand sides are Lebesgue integrable, and the successive Picard approximations for the variables will converge uniformly to $x_1, \dots, x_k$. Thus the estimates previously obtained for $|x_p|$ and $|x - S'_{p-1}|$—the error of the p-th approximation** —will be valid almost everywhere on intervals $t \sim 1/\varepsilon$.

Remark 2. The results of this paper are readily extended to the case in which ε enters the right-hand side of (1) nonlinearly and the function $X(t,x,\varepsilon)$ is continuously differentiable $(k-1)$ times with respect to ε.

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CITED LITERATURE

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* According to (34), $(x - S_{k-1}) \to 0$ as ε^k for $\varepsilon \to 0$ and fixed k, i.e., the series (7) represents the solution (1) asymptotically.

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Note: Figure translations are in progress. See original paper for figures.