ON SOME NONLINEAR PROBLEMS IN THE THEORY OF ANTENNA SYNTHESIS

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Abstract

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A. T. RAMM

ON SOME NONLINEAR PROBLEMS IN THE THEORY OF ANTENNA SYNTHESIS

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1. Introduction. There is an extensive literature devoted to the problem of antenna synthesis ([2–4]). However, as far as we know, nonlinear problems of antenna synthesis have not been studied in the literature, although the usefulness of this kind of problem is clear (see, for example, ([5]), p. 306). In the present paper a method is proposed for solving some nonlinear problems of synthesis theory. In §§ 2-4 the synthesis of bearing characteristics is studied. A method is indicated for finding the current from a given characteristic. In § 5 an optimization problem is considered: how to find a bearing characteristic with the greatest steepness at the origin of coordinates for a given level of side lobes. In § 6 the possibilities of the method for solving other nonlinear problems of synthesis theory are discussed. In § 7 it is explained in which cases the synthesis problem is approximately solvable with a prescribed accuracy.

2. The mathematical formulation of the problem is as follows. A nonlinear integral equation of the first kind is given:

$$F^2(k-k_0) - F^2(k+k_0) = g(k), \tag{1}$$

where $0 < k_0 < \pi/2$ is a certain constant number,

$$F(k) = F_j(k) = \int_{-\pi}^{\pi} j(x)\exp(ikx)\,dx, \tag{2}$$

$j(x)$ is the current distribution (it is to be found), and $g(k)$ is a given function—the bearing characteristic. It is required to determine: a) whether the solution of equation (1) is unique; b) for what functions $g(k)$ equation (1) is solvable; c) how to find the current $j(x)$ from the function $g(k)$.

3. We shall prove the uniqueness of the solution of equation (1) by contradiction. We shall assume the current distribution $j(x)$ to be a function from $L_2(-\pi,\pi)$. Suppose that there exist two solutions $j(x)$ and $\psi(x)$ of equation (1). Then

$$F_j^2(k+k_0) - F_j^2(k-k_0) = F_\psi^2(k+k_0) - F_\psi^2(k-k_0). \tag{3}$$

Introduce the notation

$$\Phi(k) = F_j^2(k) - F_\psi^2(k). \tag{4}$$

It follows from equality (2) that $\Phi(k)$ is an entire function of exponential type $\leq 2\pi$, $\Phi(k) \subset W_{2\pi}$, i.e. $\int_{-\infty}^{\infty} |\Phi(k)|^2 dk$. (For the definition of the class $W_{2\pi}$, see ([1]), p. 179, or ([2]).) Equality (3), taking (4) into account, assumes the form

$$\Phi(k+k_0) = \Phi(k-k_0). \tag{5}$$

Therefore $\Phi(k)$ is a periodic function with period $2k_0$. We shall use a lemma of S. N. Bernstein (([1]), $p.$226), from which it follows that

$$\Phi(k) = \sum_{|n|\leq[2k_0]} c_n \exp\left(i\frac{\pi n}{k_0}k\right), \tag{6}$$

where $[x]$ is the integer part of x, and c_n are constants. But $\Phi(k) \subset W_{2\pi}$; consequently, all $c_n = 0$, $\Phi(k) = 0$. Therefore

$$F_j^2(k) = F_\psi^2(k), \qquad F_j(k) = \pm F_\psi(k). \tag{7}$$

Hence $j(x) = \pm\psi(x)$. As is seen from formulas (1), (2), the sign of the function $j(x)$ has no effect on the radiation characteristic (the functions $j(x)$ and $-j(x)$ generate one and the same radiation characteristic). Therefore we shall agree not to regard the distributions $j(x)$ and $-j(x)$ as different. In view of what has been said, we summarize the proved assertion.

Theorem 1. The synthesis problem (1) has at most one solution $j(x) \in L_2(-\pi, \pi)$.

4. Let us investigate the question of for which functions $g(k)$ problem (1) is solvable in the class of functions $j(x) \in L_2$ and how to find the solution. We shall restrict ourselves to the case of even distributions $j(x) = j(-x)$. In this case

$$F(k) = F(-k).$$

Consequently, the following condition is necessary for solvability of (1).

Condition A. The characteristic $g(k)$ must be an even entire function of exponential type $\leq 2\pi$, belonging to the class $W_{2\pi}$.

We shall assume that condition A is fulfilled. Then, by the Wiener–Paley theorem, there exists an even function $\tilde{g}(x) \subset L_2$ such that

$$g(k) = \int_{-2\pi}^{2\pi} \tilde{g} \exp(ikx)\, dx. \tag{8}$$

Introduce the function $\Phi(k) \subset W_{2\pi}$

$$\Phi(k) \equiv F^2(k).$$

By the Wiener–Paley theorem

$$\Phi(k) = \int_{-2\pi}^{2\pi} \varphi(x) \exp(ikx)\, dx, \qquad \varphi \in L_2. \tag{9}$$

Write equation (1) in the form

$$2i \int_{-2\pi}^{2\pi} \varphi(x) \sin k_0 x\, \exp(ikx)\, dx = \int_{-2\pi}^{2\pi} \tilde{g}(x) \exp(ikx)\, dx. \tag{10}$$

Hence

$$\varphi(x) = \tilde{g}(x)/2i \sin k_0 x. \tag{11}$$

If the function standing on the right-hand side of equality (11) belongs to the class $L_2(-2\pi, 2\pi)$, then we find

$$F^2(k) = \int_{-2\pi}^{2\pi} \frac{\tilde{g}(x)}{2i \sin k_0 x} \exp(ikx)\, dx. \tag{12}$$

Since $F(k)$ is an entire function, all zeros of the function (12) must be of even order. If this is satisfied, then one can extract the root from both sides of equality (12) and obtain the entire function

$$F(k) = \left\{ \int_{-2\pi}^{2\pi} \frac{\tilde{g}(x)}{2i \sin k_0 x} \exp(ikx)\, dx \right\}^{1/2}. \tag{13}$$

Knowing $F(k)$, it is not difficult to find the current

$$j(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \exp(-ikx)\, dk. \tag{14}$$

We summarize the result obtained.

Theorem 2. For the solvability of equation (1) in the class of even distributions $j(x) \in L_2$, condition A is necessary. If this condition is fulfilled, then equation (1) is solvable if and only if the function (11) belongs to L_2 and the function (13) is entire. The solution is found from (14).

Remark 1. It is obvious that the function (13) will be entire if and only if all zeros of the function (12) have even order.

5. Consider the following optimization problem. It is required to find the bearing characteristic with the greatest steepness at the origin, for a given sidelobe level, or, equivalently, to find the characteristic with a given steepness and a minimal sidelobe level. Since the characteristic $g(k)$ is a function of the class $W_{2\pi}$, the matter reduces to finding, among the functions $g(k)$ satisfying the conditions of Theorem 2 and the condition

$$g'(0) = a, \qquad a = \text{const}, \tag{15}$$

that one for which*

$$\max_{-\infty<k<\infty} |g(k)| = \min_{\varphi} \max_{-\infty<k<\infty} |\varphi(k)|. \tag{16}$$

In [1, p. 355] it is proved that the solution of the posed problem in the class of all entire functions of exponential type $\leq 2\pi$ has the form

$$\varphi_0(k) = a \sin 2\pi k/2\pi. \tag{17}$$

It turns out that the function (17) is extremal also in the class of all entire functions of exponential type $\leq 2\pi$, under the condition that the right-hand side of equality (16) has the form

$$\min_{\varphi} \max_{k>d,\, k<-d} |\varphi(k)|, \quad \text{where } 0 < d < 1/4 \quad ([1], p.364).$$

It follows from this that if, in the class $W_{2\pi}$, one seeks a bearing characteristic with a given steepness, determined by equality (15), and with a minimal sidelobe level for $|k| > d$, then we arrive at the same characteristic that we would obtain if, under the same conditions, we sought a characteristic $g(k)$ for which condition (16) is fulfilled. The function (17) does not satisfy all the conditions of Theorem 2: it is not square-integrable on the real axis, and its zeros have multiplicity one. Therefore it is unrealizable by means of current distributions belonging to L_2. But by means of point radiators one can realize the characteristic (17). Namely, if in formula (2) one takes

$$j(x) = \frac{A}{2i}[\delta(x-\pi) - \delta(x+\pi)], \tag{18}$$

then $F(k) = A \sin k\pi$. The left-hand side of equality (1) takes the form

$$A^2[\sin^2(k+k_0)\pi - \sin^2 \pi(k-k_0)] = A^2 \sin 2\pi k_0 \sin 2\pi k. \tag{19}$$

The right-hand side of (19), for $A = \sqrt{a}/2\pi \sin 2\pi k_0$, coincides with the function (17), if $2k_0$ is not an integer, which can always be assumed fulfilled, since otherwise k_0 can be changed. If, instead of the unrealizable delta functions in formula (18), one uses approximating functions $\delta_n(x)$, for example,

$$\delta_n(x) = \begin{cases} 1/2\varepsilon, & \text{for } |x| \le \varepsilon, \\ 0, & \text{for } |x| > \varepsilon, \end{cases}$$

then realizable characteristics will be obtained, differing little for $|k| < k_b$ from the optimal (but unrealizable) function (17).

6. We indicate some nonlinear synthesis problems that can be solved by the proposed method. If nonlinear processing of the signal is performed by the formula

$$[F(k+k_0)]^n - [F(k-k_0)]^n = g(k), \qquad n > 0, \tag{20}$$

then the scheme of investigation given in §§ 2–4 remains valid. Let us note that increasing n is accompanied by the following circumstances. The characteristic $g(k) \subset W_\sigma$, $\sigma = n\pi$. In the class of even current distributions the characteristic $g(k)$ is an odd function. The function

$$\int_{-n\pi}^{n\pi} \frac{\tilde{g}(x)}{2i \sin k_0 x} \exp(ikx)\, dx$$

must be the n-th power of an entire function of class W_π. As n increases, the class $W_{n\pi}$ expands, but the requirement that the function

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* Condition (16) means that among all functions $\varphi \in W_{2\pi}$, the function $g(k)$ has the minimal maximum modulus (see also the paragraph following formula (17)).

$$\int_{-n\pi}^{n\pi} \frac{\tilde{g}(x)}{2i \sin k_0 x} e^{ikx}\, dx$$

must be the n-th power of a function of class W_π, narrows the class of realizable patterns. Increasing n makes it possible, while preserving the steepness at zero, to reduce the sidelobe level. In fact, in the class W_σ the solution of the extremal problem (16) under condition (15) is given by the formula (([1]), p. 355):

$$\varphi_0(k) = \frac{a}{\sigma} \sin \sigma k. \tag{21}$$

Therefore, for $\sigma = n\pi$, increasing n leads to a decrease in

$$\max_{-\infty<k<\infty} |\varphi_0(k)|$$

for an unchanged value $\varphi_0'(0) = a$. In addition to problem (20), the same method can be used to consider the more general problem:

$$\Psi(F(k+k_0)) - \Psi(F(k-k_0)) = g(k), \tag{22}$$

where the function $\Psi(z)$ is such that $\Psi(F(k))$ is an entire function of exponential type. It would be interesting to carry out the solution of problem (1) according to the scheme of work ([7]).

7. If the function $g(k)$ does not satisfy the conditions of Theorem 2, but is odd, then problem (1), while not being exactly solvable, may admit an approximate solution acceptable from the practical point of view. It is sufficient to approximate on the interval $\Delta\{-k_b < k < k_b\}$, where k_b is a sufficiently large number, the function $g(k)$ by a function $g_n(k) \subset W_{2\pi}$ satisfying the conditions of Theorem 2. It is always possible to construct an odd function $g_n(k) \subset W_{2\pi}$, approximating, with prescribed accuracy, the given odd function $g(k)$ (for example, by the method of work ([6])). In order that the function $g_n(k)$ satisfy the remaining conditions of Theorem 2, it is necessary and sufficient that the function

$$\Phi_n(k) = \int_{-2\pi}^{2\pi} \frac{\tilde{g}_n(x)}{2i \sin k_0 x} \exp(ikx)\, dx. \tag{23}$$

not vanish for $-k_b < k < k_b$, or have on this interval zeros only of even order. Indeed, if this condition is fulfilled, then there exists a continuous function $\sqrt{\Phi_n(k)}$, which can be approximated arbitrarily accurately on the interval Δ by a function $F_n(k) \subset W_\pi$. Then the function $\Phi_n(k) \subset W_{2\pi}$ will be approximated with the desired accuracy by the function $F_n^2(k) \subset W_{2\pi}$ and, consequently, by one satisfying the conditions of Theorem 2 (the zeros of the function $F_n^2(k)$ have even order).

The current distribution producing a pattern that differs little from $g_n(k)$, and therefore also from $g(k)$, is found from the formula

$$j_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_n(k) \exp(-ikx) \, dk. \tag{24}$$

8. Results analogous to those presented in §§ 2–4 have been obtained by us for the problem of synthesizing a bearing characteristic in the case of a planar aperture.

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Note: Figure translations are in progress. See original paper for figures.