UNIFORM STRUCTURES OF BORDERINGS

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Abstract

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A finite system of open sets U_i is called a bordering of a completely regular space X if the set $X \setminus \bigcup_i U_i$ is bicompact. The concept of a bordering was introduced by Yu. M. Smirnov ([2,3]) for the characterization of spaces possessing bicompact extensions whose remainders have a prescribed finite dimension. Yu. M. Smirnov showed that (under certain restrictions) a completely regular space X possesses a bicompact extension Y, for which $\dim(Y \setminus X) \leq n$, if and only if there is on it a structure of borderings of multiplicity $\leq n+1$ possessing the base property (see below).

We shall call a system of borderings Ω of a completely regular space X a uniform structure of borderings if Ω satisfies the following conditions:

1. If $\xi \in \Omega$ and $\eta < \xi$ (i.e., every set of ξ is contained in some set of η), then $\eta \in \Omega$.
2. If $\xi \in \Omega$ and B is a bicompact subset of X, then the restriction of ξ to the subspace $X \setminus B$ is contained in Ω.
3. For any $\xi, \eta \in \Omega$ there is a $\zeta \in \Omega$ such that $\xi \wedge \eta < *\zeta$ ($\xi \wedge \eta$ is the system of sets of the form $U \cap V$, $U \in \xi$, $V \in \eta$; the symbol $\xi < *\eta$ means that for every point $x \in X$ there exists a $U \in \xi$ such that the star $O_\eta x$ of the point x with respect to the system of sets η is contained in U).
4. For every point $x \in X$ and every neighborhood Ox of it there exists a bordering $\xi \in \Omega$ such that every $U \in \xi$ whose closure $[U]$ contains x is contained in Ox.

A system of borderings satisfying conditions 3, 4 will be called the confinal part of a (corresponding) uniform structure of borderings. It is easy to verify that conditions 3, 4 determine the structure uniquely. We note that in the case of open coverings, conditions 1, 3, and 4 may be taken as the basis of the definition of the uniform structure of a topological space (for the definition of the uniform structure of open coverings, see, for example, ([1])). It is not hard to verify that condition 4 is equivalent to the following requirement.

4′. For every point $x \in X$ and every neighborhood Ox of it there exists a neighborhood O_1x and a bordering $\xi \in \Omega$ such that $O_\xi O_1 x \subset Ox$.

A system of borderings satisfying conditions 3 and 4′ is called by Yu. M. Smirnov a structure of borderings possessing the base property; thus, Yu. M. Smirnov' s structure of borderings coincides with the confinal part of a uniform structure of borderings in the sense indicated above.

The set of all uniform structures of borderings of the space X is partially ordered: as usual, we regard $\Omega < \Omega_1$ if the system Ω is contained in Ω_1. The aim of the present note is to describe the partially ordered set of all uniform structures of borderings of an arbitrary completely regular space X.

Let Ω_0 be the subsystem in Ω consisting of all coverings $\alpha \in \Omega$.

Lemma 1. *For every uniform structure of borderings Ω, the system of coverings Ω_0 is a uniform structure of the space X.*

From this lemma there obviously follows

Corollary. The correspondence $\omega : \Omega \to \Omega_0$ defines an order-compatible mapping of the partially ordered set of all uniform structures of fringes of the completely regular space X onto the set of all uniform structures of finite open coverings of the space X.

We shall say that the uniform structure of fringes Ω is compatible with the uniform structure of coverings Ω_0 if $\Omega_0 = \omega(\Omega)$. Let Y be the bicompact extension of the space X corresponding to the uniform structure Ω_0 (see (1)). For every open set U of the space X, denote by $O(U)$ the set $Y \setminus [X \setminus U]$. Obviously, $O(U) \cap X = U$; $O(U)$ is maximal among all open sets in Y that cut U on X. Denote by $O_N(U)$ the intersection of $O(U)$ with the remainder $N = Y \setminus X$. From Lemma 1 it follows easily (see below) that for every fringe ξ from the uniform structure Ω, compatible with Ω_0, the system of sets $O_N(U)$, $U \in \xi$, is a covering of the remainder N. Denote by Σ the system of all coverings of N obtained in this way from fringes of the structure Ω, and by Σ_0 the uniform structure of the space N obtained in the same way from Ω_0.

Theorem. The correspondence $\sigma : \Omega \to \Sigma$ carries out an isomorphism of the set of all uniform structures of fringes of the completely regular space X, compatible with the structure of finite open coverings Ω_0, onto the set of all uniform structures of finite open coverings of the remainder N that follow the uniform structure Σ_0. If N is dense in Y (i.e., if X has no points of local bicompactness), then σ is a mapping onto the whole set of uniform structures of coverings of N that follow Σ_0. Moreover, in this case the operator $O_N(\ )$ carries out an isomorphism of the set of all uniform structures of coverings of the space X that follow Ω_0, onto the set of all uniform structures of fringes of N compatible with the structure Σ_0. The operator $O_X(V)$, $V \subset N$, gives the inverse isomorphisms.

Thus the concept of a uniform structure of fringes is not only a generalization of the concept of a uniform structure of coverings, but also, in a certain sense, a dual concept.

We now pass to the proof of the formulated assertions. The following proposition is a strengthening of condition 4.

Lemma 2. For any bicompact set $B \subset X$ and for any neighborhood W of the set B, there is a fringe $\xi \in \Omega$ such that every $U \in \xi$ whose closure meets B is contained in W.

Proof. For every point $x \in B$ choose a fringe $\xi(x) \in \Omega$ such that $U \subset W$, if $x \in [U]$, $U \in \xi(x)$. Obviously, the point x has a neighborhood $B(x)$ in B such that from $[U] \cap B(x) \neq \emptyset$, $U \in \xi(x)$, it follows that $U \subset W$. From the covering $\{B(x)\}$ of the bicompact B choose a finite subcovering $B(x_1), \dots, B(x_n)$. By conditions 1, 3 the fringe $\xi = \xi(x_1) \wedge \dots \wedge \xi(x_n)$ belongs to Ω. The fringe ξ satisfies the requirement of the lemma.

Proof of Lemma 1. For every fringe $\xi \in \Omega$ denote by B_ξ the bicompact $X \setminus \bigcup_i U_i$, $U_i \in \xi$. By condition 1, the covering α that includes ξ and any finite collection of open sets covering B_ξ belongs to Ω_0; therefore the system Ω_0 is nonempty. The system Ω_0 obviously satisfies condition 1. Condition 4 is also fulfilled for Ω_0: first choose a fringe $\xi \in \Omega$ satisfying this condition, and then add to it the set Ox and any open set W containing $B_\xi \setminus Ox$ but whose closure does not contain the point x; the resulting covering is contained in Ω_0 and satisfies the needed requirements. By condition 3 the covering $\alpha \wedge \beta$ belongs to Ω_0, if $\alpha, \beta \in \Omega_0$. Therefore it remains only to prove that for any $\alpha \in \Omega_0$ there exists $\beta \in \Omega_0$ such that $\alpha <_* \beta$.

Let $\alpha \in \Omega_0$ be an arbitrary cover. In accordance with condition 3, there is a $\xi \in \Omega$ such that $\alpha < *\xi$. Let α' be a cover of the Čech extension βX, consisting of the sets $O(U)$, $U \in \alpha$, and of the set $\beta X \setminus B_\xi$; let γ be a finite open cover of βX, star-refined into α', and let $\bar{\gamma}$ be a closed cover of βX refined into γ. Let $\bar{B}_1, \dots, \bar{B}_k$ be those sets of $\bar{\gamma}$ which have nonempty intersection with B_ξ, and let $\Gamma_1, \dots, \Gamma_k$ be elements of γ containing them; denote $\bar{B}_i \cap B_\xi$, $\Gamma_i \cap X$, respectively, by B_i, V_i. The sets B_i cover B_ξ, and the system of sets $\nu = \{V_i\}$ is star-refined into α.

By Lemma 2, for each B_j there is a border $\xi_j \in \Omega$ such that from $[U] \cap B_j \neq \emptyset$, $U \in \xi_j$, it follows that $U \subset V_j$. Let $\eta \in \Omega$ be the border equal to the intersection of all the borders $\xi, \xi_1, \dots, \xi_k$. From the fact that $U \in \eta$ and $[U] \cap B_\xi \neq \emptyset$, it follows, obviously, that $U \subset V_j$ for some j. Let C be the closure of the union of all those $U \in \eta$ whose closures do not intersect B_ξ; let ν' be the system of sets $V_j' = V_j \setminus C$, and let $W = \bigcup_{j=1}^k V_j'$. By construction we have $\alpha < *(\nu' \cup \eta)$; however, the system of sets $\nu' \cup \eta$ is not a cover (the points of the set $B_\eta \setminus W$ are not covered).

Let W_1 be a neighborhood of the set B_ξ such that $[W_1] \subset W$. Denote by ξ' the restriction of the system ξ to $X \setminus [W_1]$, by ν'' the restriction of ν' to W_1, and by ξ'' the intersection with ν' of the restriction of the system ξ to the set $W \setminus B_\xi$. Put $\beta = \nu'' \cup \eta \cup \xi' \cup \xi''$. Obviously, β is a finite open cover of X. Since $\beta < \eta$ and $\eta \in \Omega$, we have $\beta \in \Omega_0$. For every point $x \in W_1$, the star $O_{\beta x}$ consists of sets contained in sets of the system ν, and for every point $x \in X \setminus W_1$, of sets contained in sets of the system ξ. Therefore $\alpha < *\beta$. The lemma is proved.

It follows from this lemma that, for any $\xi \in \Omega$, the system of sets

$$O_N(\xi) = \{O_N(U),\, U \in \xi\}$$

covers $N = Y \setminus X$. Indeed, let y be an arbitrary point of N, and let O_y be a neighborhood in Y whose closure does not intersect B_ξ. Then the cover of X consisting of the elements of ξ and of the set $X \setminus [O_y \cap X]$ belongs to Ω_0 and, consequently, extends to a cover of Y, and moreover y, obviously, does not belong to $O(X \setminus [O_y \cap X])$.

Proof of the theorem. We begin with some auxiliary observations. Recall that an open set U is called canonical if

$$U = \mathrm{Int}[U] = X \setminus [X \setminus [U]].$$

We shall call a border $\xi \in \Omega$ canonical if every $U \in \xi$ is a canonical open set of the space $X \setminus B_\xi$. For every $\xi \in \Omega$, denote by $\bar{\xi}$ the border of X obtained by replacing each $U \in \xi$ by the minimal canonical open (in $X \setminus B_\xi$) set $\bar{U}$ containing U. Since $\bar{\xi} < \xi$, we have $\bar{\xi} \in \Omega$. If $\xi < *\eta < *\zeta$, then $\bar{\xi} < \bar{\zeta}$ (since the cover ζ of the space $X \setminus B_\zeta$ is refined by ξ). Therefore the subsystem $\overline{\Omega} \subset \Omega$, consisting of all canonical borders, is a cofinal part of the structure Ω. Obviously, if $\xi \in \overline{\Omega}$, then for any bicompact set $B \subset X$ the restriction of ξ to $X \setminus B$ also belongs to $\overline{\Omega}$.

For every $U \in \xi$, $\xi \in \Omega$, the set $O(U)$, being the maximal open set cutting out U on X, is a canonical open set in $Y \setminus B_\xi$. If η is the restriction of ξ to $X \setminus B$, where B is a bicompact set, then $O_N(\eta) = O_N(\xi)$. Further, the relations $\xi < \eta$, $\xi < *\eta$ imply, obviously, the relations $O_N(\xi) < O_N(\eta)$, $O_N(\xi) < *O_N(\eta)$ (ξ, η arbitrary). The system of covers

$$O_N(\Omega) = \{O_N(\xi),\, \xi \in \Omega\}$$

satisfies condition 4, since this condition is satisfied by the system of covers $O_N(\overline{\Omega}_0)$ contained in it. For every uniform structure of covers Ω_0', for which $\Omega_0 < \Omega_0'$, $O_N(\overline{\Omega}_0')$ is a system of borders of the remainder N, also, obviously, satisfying condition 4.

Let N be dense in Y. Then for every open $U \subset X$ we have the inclusion

$$U \subset O_X(O_N(U)).$$

Moreover, if $U \in \xi$, $\xi \in \overline{\Omega}$, then $U =$

$= O_X(O_N(U)) \cap (X \setminus B_\xi)$. If, however, ξ is a cover of X consisting of a finite number of canonical sets, then $\xi = O_X(O_N(\xi))$. Taking into account that it suffices to consider only canonical covers and bordifications, we obtain the assertion of the theorem for the case when N is dense in Y.

To prove Theorem 1 it remains to consider the special case when Y is not equal to the closure of N. We have seen that the system of covers $O_N(\Omega)$ of the set N satisfies conditions 3 and 4. Therefore it is the cofinal part of some uniform structure of finite covers of N (obviously following Σ_0), which we shall put in correspondence with the structure of bordifications Ω. The fact that different bordification structures of X will correspond to different cover structures of N follows from the following assertion: from $O_N(\gamma) < O_N(\xi)$ and $\xi \in \Omega$ it follows that $\gamma \in \Omega$ (γ is any bordification). For the proof take bordifications $\eta, \zeta \in \Omega$ such that $\xi < *\eta < *\zeta$. Let $Y' = Y \setminus (B_\gamma \cup B_\xi)$, and let γ', ξ', ζ' be covers of Y' obtained by extending to Y' the bordifications γ, ξ, ζ. For any $U' \in \xi'$ choose $V' \in \xi'$ such that $[U'] \subset V'$ (closure in Y'), and $W' \in \gamma'$ such that $V' \cap N \subset W' \cap N$. The set $[U'] \setminus W'$, closed in Y', does not intersect N (otherwise the set $V' \cap N$ would not be contained in $W' \cap N$). Summing the sets of the form $[U'] \setminus W'$, constructed in this way for all $U' \in \xi'$, we obtain a set C, closed in Y', which does not intersect N. Let $B = B_\gamma \cup B_\xi \cup C$; the set B is bicompact and lies in X. The restriction of the bordification ζ to $X \setminus B$ is inscribed in γ. Therefore, by conditions 1, 2, $\gamma \in \Omega$.

Remark 1. If the extension Y has a countable base, then $O_N(\ )$ maps the uniform bordification structures compatible with Ω_0 onto all uniform structures of finite covers of N following Σ_0. It can be shown that in the general case this is not so.

Remark 2. The assertion of Lemma 1 is valid for a system Ω of arbitrary (not necessarily finite) bordifications satisfying conditions 1, 2, 3, and 4′; in the proof, however, one must make the obvious changes caused by replacing requirement 4 by condition 4′ of Yu. M. Smirnov (for example, in Lemma 2 there will be found such a $\xi \in \Omega$ and such a neighborhood $V \supset B$ that every $U \in \xi$ for which $U \cap V \neq \emptyset$ is contained in W).

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Note: Figure translations are in progress. See original paper for figures.