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Abstract

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PROBLEMS OF S. L. SOBOLEV TYPE IN THE CASE OF SUBMANIFOLDS WITH MULTIDIMENSIONAL SINGULARITIES

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1. Introduction. By problems of S. L. Sobolev type we mean problems for (pseudo-) differential equations in which the boundary conditions are prescribed on a submanifold that is not the boundary of the principal manifold. In the paper ([1]) the theory of such problems was developed for smooth (without singularities) submanifolds. The study of problems of S. L. Sobolev type in the case of submanifolds with singularities was begun in the note ([2]), where submanifolds with isolated point singularities were studied.

In the present note we assume that the submanifold on which the boundary conditions are prescribed has multidimensional singularities.

The main result of the work is a finiteness theorem (normal solvability) for the problems indicated in the title. To such problems we apply the principle of locality, and therefore the center of gravity in establishing the finiteness theorem falls on the study of a model—the localization of the problem at an arbitrary point of the singular submanifold. In other words, one must consider the operator generated by homogeneous (pseudo-) differential expressions with coefficients fixed at an arbitrary point of the singular submanifold. After a Fourier transform along the singular submanifold, we arrive at a family of operators parametrized by the cotangent space to the singular submanifold. Analytically it is shown that outside the zero section this family is a family of Fredholm operators.

Adding now the required number of boundary and coboundary operators, we can associate with the resulting Fredholm family a family of isomorphisms. The corresponding global problem turns out to be Fredholm.

2. Description of the singularity. Let M be a smooth closed manifold of dimension N, and let Y be a closed submanifold of the manifold M, embedded in the following singular way. The manifold Y contains a smoothly embedded in M submanifold X, which in what follows will be called singular. The intersection of the manifold Y with the tubular neighborhood $\mathrm{Tub}(X, M)$ of the manifold X in M is a transversal intersection along the manifold X of l smooth submanifolds

Y_p, with codimensions (in M) equal to ν_p. If the manifold X is a disconnected sum of some number of smooth manifolds, then the preceding assumption applies to each component of this sum.

3. Local situation. In this section we shall assume that the manifold M is the N-dimensional vector space R^N with coordinates $(x,t) = (x^1, \dots, x^n, t^1, \dots, t^\nu)$, where the singular submanifold X is the n-dimensional subspace R^n and is given by the equations $t^1 = \dots = t^\nu = 0$, or briefly $t = 0$. The submanifold Y is the union of l hyperplanes transversally intersecting along the hyperplane R^n.

Denote by $\Gamma^s(R^N)$ (s is a real number) the C. L. Sobolev space of distributions on R^N with norm

$$\|f\|_s^2 = \iint |(1+\Delta)^{s/2} f|^2 \, dx \, dt.$$

Here

$$\Delta = -\left(\sum_{i=1}^{\nu} \frac{\partial^2}{\partial t_i^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \right)$$

is the positive Laplace operator.

Let, further,

$$D : \Gamma^s(R^N) \to \Gamma^{s-m}(R^N)$$

be an elliptic scalar* pseudodifferential operator of real order m with symbol independent of x and t.

Our aim is to study the solvability of the comparison

$$Du \equiv f \pmod{Y}, \tag{1}$$

i.e., of an equation to be satisfied up to distributions concentrated on the submanifold Y.

In doing so, we shall seek solutions in the space $\Gamma^s(R^N)$, while the right-hand sides in (1) belong to the space $\Gamma^{s-m}(R^N)$.

Define the nonnegative numbers χ_p by setting

$$\chi_p = \begin{cases} [m - s - \nu_p/2], \\ m - s - \nu_p/2 - 1. \end{cases} \tag{2}$$

Here the square brackets denote the integer part of a number, and the value $m - s - \nu_p/2$ is taken in the case when the expression in square brackets is not an integer, while the value $m - s - \nu_p/2 - 1$ is taken in the opposite case.

Denote by $B^Y_{j,p}$ mappings which are the composition of pseudodifferential operators B_{jp} of order b_{jp} and the restriction operator d^Y_p to the submanifold Y_p.

For each $p = 1, \ldots, l$ denote by l_p the number of operators $B_{j,p}$. We note that the number l_p is uniquely determined by the number χ_p (see ([1])). The totality of operators $\{B^Y_{j,p}\}$, $j \le l_p$, $p \le l$, will for brevity be denoted by B^Y, or even simply by B in places where this will not cause confusion. The operator $B^Y = B$ will be called the Y-boundary operator.

We note that for $s > \max_{j,p}(b_{jp} + \nu_p/2)$ the mappings $B^Y_{j,p}$ define a continuous operator.

Assume that at each point of the hyperplane Y_p, $p = 1, \ldots, l$, the operators (D, B) are connected by an algebraic condition (elly) of the Shapiro–Lopatinskii type ([1]).

With the "pair" (D, B) one can associate the continuous mapping

$$(D, B)^s : \Gamma^s(R^N) \to \Gamma^{s-m}(R^N)/\Delta(Y) \oplus \Gamma^{s-b_{jp}-\nu_p/2}(Y_p). \qquad (3)$$

Here $\Delta(Y)$ denotes the subspace of the space $\Gamma^{s-m}(R^N)$ of elements concentrated on the submanifold Y.

Let us pass in the operator (3) to the Fourier transform with respect to the variables

$$x = (x^1, \ldots, x^n)$$

(along the special submanifold).

Let $\xi = (\xi_1, \ldots, \xi_n)$ be the vector in the Fourier-transform space dual to the vector $x = (x^1, \ldots, x^n)$. Then for each fixed $\xi \ne 0$ we obtain the operator

$$(D, B)^s_\xi : \Gamma^s(R^\nu) \to \Gamma^{s-m}(R^\nu)/\Delta(Y/R^n) \oplus \bigoplus_{j,p} \Gamma^{s-b_{pj}-\nu_p/2}(Y_p). \qquad (4)$$

In other words, we obtain a family of operators parametrized by the space R^n. Our aim is to prove a theorem

* The matrix case is considered, in principle, analogously.

finiteness for each of the operators of this family, corresponding to a nonzero vector ξ, i.e., the establishment of the Fredholm property of such operators. To solve this problem, continuous operators are constructed which reduce the operator $(D,B)^s_\xi$ $(\xi \neq 0)$ to compact operators. The construction of such operators is carried out with the aid of a certain special (finite) partition of unity of the space R^ν. Among the elements of the partition of unity there are the following three types: elements not containing points of the submanifolds Y_p; elements containing points of the submanifold Y_p, but not containing the origin of coordinates; and, finally, an element containing the origin of coordinates. The operator considered in neighborhoods of the first two types is continuously invertible.

In a neighborhood of the origin we can discard the lower-order terms in the operator (4), i.e., set $\xi = 0$ in the operator (4). Then we obtain an operator with homogeneous symbol. This operator, in the space $\widetilde{R}^\nu$ dual (with respect to the Fourier transform), is equivalent to a matrix operator whose entries are integral operators. The matrix operator, after a modification of the Mellin transform, is algebraized and thus admits a complete investigation. In particular, one can show that for all numbers s from the corresponding interval, except for a certain set of isolated numbers, which we shall hereafter call exceptional, the operator $(D,B)^s_0$ is an invertible operator up to an infinitely smoothing operator.

These considerations make it possible to establish the following important

Proposition. *The operator $(D,B)^s_\xi$ is Fredholm if and only if the number s is not exceptional.*

Remark. The exceptional numbers s are the real parts of the poles of a certain meromorphic function. In the case when $\operatorname{codim}(X, Y_p) = 1$ for all $p = 1, \dots, l$, this function is constructed explicitly.

It follows from the proposition that for nonexceptional s the index of the operator $(D,B)^s_\xi$ is defined:

$$\operatorname{Index}(D,B)^s_\xi = \dim\ker(D,B)^s_\xi - \dim\operatorname{coker}(D,B)^s_\xi = \alpha(\xi,s) - \beta(\xi,s).$$

By virtue of the homotopy invariance of the index, it does not depend on the vector $\xi \neq 0$. Moreover, one can show that the operators $(D,B)^s_\xi$ for different $\xi \neq 0$ are unitarily equivalent; consequently, each of the numbers $\alpha(\xi,s)$ and $\beta(\xi,s)$ also does not depend on the vector ξ: $\alpha(\xi,s) = \alpha(s)$, $\beta(\xi,s) = \beta(s)$.

We now add to the operator $(D,B)^s_\xi$ $\alpha(s)$ boundary operators B_ξ and $\beta(s)$ coboundary (3) operators G_ξ, subject to the natural condition of the Shapiro-Lopatinskii type (condition ell_X), in such a way that the operator $(D,B)^s_\xi$, endowed in this way, realizes an isomorphism of the spaces

$$(D,B^Y,B_\xi,G_\xi)^s_\xi : C^{\beta(s)} \oplus \Gamma^s(R^\nu) \to (\Gamma^{s-m}(R^\nu)/\Delta) \oplus \sum_{p,j} \Gamma^{s-b^Y_{p,j}-\nu_p/2}(Y_p) \oplus C^{\alpha(s)}. \tag{5}$$

If we make the inverse Fourier transform (from ξ to x), then the operators B_ξ and G_ξ pass respectively into boundary and coboundary operators, which we denote by B^X, G^X. We shall say of these new operators that they satisfy condition (ell_X), if the operator (5), constructed with the aid of the operators B^X and G^X, is an isomorphism.

4. The global situation. Let now M be a smooth manifold, Y its submanifold, embedded in the special manner discussed above, and X a special submanifold. By $\Gamma^s(M)$ we denote the Sobolev space of distributions of order s.

Define the operator

$$(D^M,\ B^Y,\ B^X,\ G^X),$$

where

$$D^M:\ \Gamma^s(M) \to \Gamma^{s-m}(M)$$

is a pseudodifferential operator on the manifold M of order m;

$$B^Y = \{B_p^{Y_p}\}:\ \Gamma^s(M) \to \bigoplus_{p,j} \Gamma^{s-b_p^Y-\nu_p/2}(Y_p)$$

is a column of Y-boundary operators of orders b_p^Y;

$$B^X = \{B_k^X\}:\ \Gamma^s(M) \to \bigoplus_{k} \Gamma^{s-b_k^X-\nu/2}(X)$$

is a column of X-boundary operators of orders b_k^X;

$$G^X = \{G_{lp}^X\}:\ \bigoplus_{l} \Gamma^{s_l}(X) \to \bigoplus_{p} \Gamma^{s-b_p^Y-\nu_p/2}(Y_p)$$

is a matrix of X-coboundary operators of orders g_{lp} and $s_l = s + g_{lp} - b_p - \nu_p/2$.

Definition. We shall say that the operator $(D,\ B^Y,\ B^X,\ G^X)$ is elliptic if: 1) at every point of the manifold M the operator D^M is elliptic; 2) at every point of the manifold Y the condition ell_Y is satisfied; 3) at every point of the manifold X the condition ell_X is satisfied.

In these terms the main theorem holds.

Theorem (finiteness). Let s be nonexceptional. Then the following conditions are equivalent:

A. The operator $(D,\ B^Y,\ B^X,\ G^X)$ is elliptic.

B. The operator $(D,\ B^Y,\ B^X,\ G^X)$ is Fredholm.

C. For arbitrary functions $u \in \Gamma^s(M)$, $v \in \bigoplus \Gamma^{s_l}(X)$, the inequality

$$\|u\|_s + \sum_l \|v\|_{s_l} \leq$$

$$\leq \text{const}\left(\|Du\|_{s-m} + \sum_p \|B_p^Y u\|_{s-b_p^Y-\nu_p/2} + \sum_k \|B_k^X u\|_{s-b_k^X-\nu/2} + \|u\|_{s-1} + \sum_l \|v\|_{s_l-1}\right)$$

holds.

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Note: Figure translations are in progress. See original paper for figures.