\(\Phi_-\)-OPERATORS IN LOCALLY CONVEX SPACES

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MATHEMATICS

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$\Phi\_$-OPERATORS IN LOCALLY CONVEX SPACES

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In the present note the main attention will be devoted to the investigation of the following question. Let E, F be locally convex spaces; $\varphi : E \to F$ a $\Phi\_$-operator; $k : E \to F$ a compact operator. Will $\varphi + k$ be a $\Phi\_$-operator? In other words, are $\Phi\_$-operators stable with respect to perturbations by compact operators? For the case where E and F are Fréchet spaces and the operator φ is continuous, stability was proved by L. Schwartz ([5]), Proposition 2, and by G. Köthe ([6]), Proposition 9) (in fact, the arguments of L. Schwartz are applicable to a somewhat broader class of spaces). In the case of arbitrary locally convex spaces E and F, the question remains open. Nevertheless, it is possible to prove such stability for a broad class of spaces used in analysis, and thereby to strengthen the results of L. Schwartz, G. Köthe, and F. Browder ([7]). In particular, the question is answered affirmatively for the spaces $\mathscr{D}$ and $\mathscr{D}'$. In addition, the note formulates a number of propositions characterizing $\Phi\_$-admissible perturbations. For greater generality, closed (and, in general, discontinuous) $\Phi\_$-operators will be considered throughout.

By the letters E and F we shall denote locally convex spaces. The notation $\varphi : E \to F$ will mean that φ is a linear mapping from E into F, and by D_φ, N_φ, and R_φ we shall denote, respectively, the domain of definition, the null space, and the range of φ. We shall say that $\varphi : E \to F$ is open (almost open, cf. ([10])) if, for every neighborhood of zero $U \subset E$, $\varphi(U \cap D_\varphi)$ $(\overline{\varphi(U \cap D_\varphi)})$ is a neighborhood of zero in $R_\varphi (\overline{R}_\varphi)$. By $L(E, F)$ we denote the space of continuous linear operators from E into F. A closed mapping $\varphi : E \to F$ will be called a $\Phi\_(\Phi_+)$-operator ([1]) if R_φ is closed, φ is open, and $\operatorname{codim} R_\varphi < \infty$ $(\dim N_\varphi < \infty)$. An $a \in L(E, F)$ will be called a $\Phi\_(\Phi_+)$-admissible perturbation ([2]) if $\varphi + a$ is a $\Phi\_(\Phi_+)$-operator for every $\Phi\_(\Phi_+)$-operator φ. Let $\varphi : E \to F$ be given. We shall call $a : E \to F$ φ-compact (φ-precompact) if $D_a \supset D_\varphi$ and there exist neighborhoods of zero $U \subset E$ and $V \subset F$ such that $a(U \cap \varphi^{-1} V)$ is relatively compact (precompact) in F (in other words, this means that the restriction $a|_{D_\varphi} : D_\varphi \to F$ is a compact (precompact) operator when D_φ is endowed with the weakest topology for which $\varphi : D_\varphi \to F$ and the canonical embedding $D_\varphi \to E$ are continuous).

§ 1. We shall say that the pair of spaces E, F satisfies condition $*(**)$ if, for every $\Phi\_$-operator $\varphi : E \to F$ and every continuous φ-precompact (φ-compact) operator $k : E \to F$, $\varphi + k$ is a $\Phi\_$-operator.

Theorem 1. Let $\varphi : E \to F$ be almost open and $\operatorname{codim} \overline{R}_\varphi < \infty$. Then $\varphi + k$ has the same properties for every φ-precompact operator $k : E \to F$. Moreover, for all $\lambda \in C$, except perhaps for a sequence $(\lambda_n)_{n \in \mathbb{N}}$ with a unique limit point at infinity,

$$\operatorname{codim} \overline{R}_{\varphi + \lambda k} = \min_{\lambda \in C} \operatorname{codim} \overline{R}_{\varphi + \lambda k}.$$

Combining Theorem 1 with Theorem 4.7 in ([10]), we obtain the following generalization of F. Browder' s theorem (([7]), Theorem 2.3):

Corollary 1. *If E is perfectly complete ([10]), and F is separable, then the pair E, F satisfies condition $(*)$.*

Using the fact that every mapping $\varphi : E \to F$ which is simultaneously weakly open and almost open is open (see ([10])), we obtain the following generalization of Proposition 2 from ([5]), formulated for weakly open mappings:

Corollary 2. *Let E and F be separable, $k \in L(E, F)$ a compact operator, and $\varphi : E \to F$ a Φ-operator satisfying the condition:*

(S) For every absolutely convex compact set $K_1 \subset F$ there is an absolutely convex compact set $K_2 \subset E$ such that $\varphi(D_\varphi \cap K_2) \supset K_1$.

Then $\varphi + k$ is a Φ-operator.

We shall need a variant of a construction due to De Wilde ([11]).

Definition 1. A **web** in the space E is a family of balanced sets $e_{n_1 \dots n_k}$ $(k, n_1 \dots n_k \in N)$ such that:

a)

$$\bigcup_{n_1=1}^{\infty} e_{n_1}$$

absorbs E;

$$\bigcup_{n_k=1}^{\infty} e_{n_1 \dots n_{k-1} n_k}$$

absorbs all elements of the set $e_{n_1 \dots n_{k-1}}$ for arbitrary $k, n_1 \dots n_k$;

b) for every sequence $(n_k)_{k \in N}$ the following condition is satisfied: if $f_k \in \operatorname{co} e_{n_1 \dots n_k}$, then the series $\sum f_k$ converges in E;

c) for any $k, n_1 \dots n_k \in N$ and $x \in e_{n_1 \dots n_{k-1}}$ there exists $n'_k \in N$ such that $e_{n_1 \dots n_{k-1} n'_k}$ absorbs x and all elements of $e_{n_1 \dots n_{k-1} n_k}$.

Remark. The definition of a web given here differs from De Wilde' s definition by the addition of condition c). Nevertheless, as is easily verified, all permanence properties listed by De Wilde (including stability under passage to countable inductive and projective limits) are preserved also under our definition. Moreover, the concrete spaces indicated in De Wilde' s paper (including Fréchet spaces, strong duals of metrizable spaces, and inductive limits of sequences of metrizable spaces) also possess a web in the sense of Definition 1.

Lemma 1. *Let the separable space F be the limit of a direct spectrum of spaces F_α (which we shall regard as vector subspaces of F), and let F_1 be a subspace in F such that, for every α, $F_1 \cap F_\alpha$ is closed in F_α (in the topology of F_α), and moreover $\dim F_\alpha / F_1 \cap F_\alpha < \infty$. Then F_1 is closed and*

$$F_1 = \varinjlim F_\alpha \cap F_1.$$

Lemma 2. *Let E and F be separable, and suppose F is represented as a strict inductive limit* ([12]) *(see also* ([15])*) of a sequence of spaces F_n. Then, if for each n the pair E, F_n satisfies condition $*(**)$, the pair E, F also satisfies condition $*(**)$.*

Definition 2. We shall say that a space F **has property (K)** if every absolutely convex compact set in F is bounded (in the sense of Definition 1 in ([13])) with respect to some absolutely convex compact set.

Theorem 2. *Let E and F be separable, and suppose E has a web. Then:*

a) *if F is a bornological space (i.e. of the second category in itself) or a strict inductive limit of bornological spaces, then the pair E, F satisfies condition $*$;*

b) *if F is a space of type (β)* ([15]) *satisfying condition (K), then the pair E, F satisfies condition $**$.*

Remark. If X is a barrelled (or, more generally, π-barrelled ([14])) space of type $\overline{S}$ ([13]), then its strong dual X' is a spac-

with a space of type (β) and satisfies condition (K). In particular, such is the space of distributions $\mathcal{D}'$.

§ 2. In the work of M. A. Gol' dman and S. N. Krachkovskii ([8]) it is shown that, for any separable topological linear spaces X and Y, the following assertion is true:

If $\varphi : X \to Y$ is open and R_φ is closed, then for any finite-dimensional $k \in L(X, Y)$ the mapping $\varphi + k$ is open and $R_{\varphi+k}$ is closed.

In the case where X and Y are locally convex spaces, considered with their weak topologies, a stronger assertion is true:

Theorem 3. Let E and F be separable, $\varphi : E \to F$. The following conditions are equivalent:

a) φ is weakly open and R_φ is closed;

b) for every finite-dimensional $k \in L(E, F)$, $R_{\varphi+k}$ is closed;

c) for every finite-dimensional $k \in L(E, F)$, $\varphi + k$ is weakly open.

Corollary. Let E and F be separable. In order that $\alpha \in L(E, F)$ be a $\Phi_+(\Phi\_)$-admissible perturbation, it is necessary and sufficient that, for every $\Phi_+(\Phi\_)$-operator $\varphi : E \to F$, the conditions

$$\dim N_{\varphi+\alpha} < \infty$$

$$(\text{codim } \overline{R}_{\varphi+\alpha} < \infty)$$

hold and that $\varphi + \alpha$ be open.

For some concrete spaces one can obtain a more precise characterization of $\Phi\_$-admissible perturbations. We begin with an assertion which is, in a certain sense, converse to Theorem 1.

Theorem 4. Let F be a space of type (β). If $\varphi : E \to F$ is not almost open, then there is a compact operator $k : E \to F$ with $D_k = D_\varphi$ such that

$$\text{codim } \overline{R}_{\varphi+k} = \infty.$$

Corollary (cf. Proposition 5b ([9])). Let E be perfectly complete, and let F be a separable space of type (β). Then:

a) in order that a closed $\varphi : E \to F$ be a $\Phi\_$-operator, it is necessary and sufficient that, for every compact $k : E \to F$ with $D_k \supset D_\varphi$,

$$\text{codim } \overline{R}_{\varphi+k} < \infty;$$

b) in order that $\alpha \in L(E, F)$ be a $\Phi\_$-admissible perturbation, it is necessary and sufficient that, for every $\Phi\_$-operator $\varphi : E \to F$,

$$\text{codim } \overline{R}_{\varphi+\alpha} < \infty.$$

Let us consider one class of $\Phi\_$-admissible perturbations. By analogy with ([3]), introduce the following

Definition 3. An $\alpha \in L(E, F)$ will be called **strictly cosingular** if there does not exist a closed subspace $F_1 \subset F$ with $\text{codim}\, F_1 = \infty$ such that $\pi\alpha$ is open and surjective. (Here $\pi : F \to F/F_1$ is the canonical mapping.)

The following assertion is true, generalizing the results of ([4]):

Theorem 5. Let E be perfectly complete, and let F be a separable space of type (β). Then:

a) $\alpha \in L(E,F)$ is strictly cosingular if and only if, for every closed subspace $F_1 \subset F$ with $\operatorname{codim} F_1 = \infty$, there exists a closed subspace $F_2 \subset F$ with $\operatorname{codim} F_2 = \infty$ such that $F_1 \subset F_2$ and $\pi_2\alpha$ is compact (here $\pi_2 : F \to F/F_2$ is the canonical mapping);

b) the totality of strictly cosingular operators acting from E into F forms a subspace in the linear space of all Φ_--admissible perturbations. This subspace is a two-sided ideal in $L(E,F)$, if $E = F$.

Remark. All the assertions of this paragraph concerning $\Phi_-(\Phi_+)$-admissible perturbations remain valid if one considers only continuous (and everywhere defined) $\Phi_-(\Phi_+)$-operators and, correspondingly, defines $\Phi_-(\Phi_+)$-admissible perturbations.

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Note: Figure translations are in progress. See original paper for figures.