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Abstract

Full Text

MATHEMATICS

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ON FINITE GROUPS OF AUTOMORPHISMS OF HYPERFINITE FACTORS

(Presented by Academician I. N. Vekua, 10 III 1969)

1. The purpose of the present article is to set forth the main points of the proof of the following theorem.

Theorem 1. *Let M be a factor of type II_1, whose elements are operators in a separable Hilbert space H. Let, further, G be a finite group of outer automorphisms of M.*

If M is an approximately finite factor, then $\mathfrak{M} = G \times M$ is likewise an approximately finite factor. The converse is true.

Since a factor of type III which is an inductive limit of hyperfinite factors is itself hyperfinite, Theorem 1 implies the following result.

Theorem 2. *Let M be an approximately finite factor of type II_1, and let G be a group of outer automorphisms of M, which can be regarded as an increasing sequence of finite groups of automorphisms of M.*

Then the factor $G \times M$ is also approximately finite.

Theorem 2 is a generalization of the theorem of Murray-von Neumann ([2]) on ergodic freely acting automorphisms of commutative weakly closed $*$-algebras.

As an example of the group G referred to in Theorem 2, one may take, for instance, the group of all permutations of a countable number of elements, where each element of this group permutes only a finite number of elements.

2. We shall set forth auxiliary results.

Lemma. *Let M be a factor of type II_1, the same as in the formulation of Theorem 1, and let G be a finite group of outer automorphisms of M.*

If M_0 is the set of all elements of M fixed with respect to G, then M_0 is a subfactor of M.

Furthermore, in order that the crossed product $\mathfrak{M} = G \times M$ be an approximately finite factor, it is necessary and sufficient that M_0 be an approximately finite factor.

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A few words concerning the proof of the lemma. M_0 consists of elements M of the form

$$\sum_{g\in G} m^g \quad (m \in M).$$

Further,

$$Q = \frac{1}{n}\sum_{g\in G} g \times 1,$$

where n is the order of the group G, is a projection from $\mathfrak{M}$. But then $Q\mathfrak{M}Q$ is a factor in the space $Q(G \otimes H)$. The factor $Q\mathfrak{M}Q$ consists of elements of the form

$$Q\sum_{g\in G}(1 \times m^g) \quad (m \in M).$$

Thus one can establish an algebraic isomorphism between M_0 and $Q\mathfrak{M}Q$. The further arguments are evident.

We now formulate the theorem whose proof is given in ([3]).

Theorem A. *Let M be a factor of type II_1, the same as in the formula-*

of Theorem 1. Let, further, G be a cyclic group of outer automorphisms of M of order n.

If M is an approximately finite factor, then $\mathfrak{M} = G \times M$ is also an approximately finite factor. The converse is true.

3. We proceed to the proof of Theorem 1. Suppose first that M is an approximately finite factor, and prove that $\mathfrak{M} = G \times M$ is also approximately finite. For this, by the lemma, it suffices to verify that M_0 is approximately finite.

Let us carry out this program. Let n be the order of G. Consider the Hilbert space H_n, $\dim H_n = n$. Denote by A_n a maximal commutative subring in H_n. Let $P_0, P_1, \dots, P_{n-1}$ be minimal projections generating A_n. Consider a group of outer automorphisms of A_n, isomorphic to G; then $P_i^g = P_j$ $(i \neq j)$.

Now, in the Hilbert space $\widetilde{H} = H \otimes H_n$, construct the ring of operators $\widetilde{M} = M \otimes A_n$ as the tensor product of the rings M and A_n. The ring $\widetilde{M}$ has a group of outer automorphisms isomorphic to the group G:

$$(m \otimes 1)^g = m^g \otimes 1; \qquad (1 \otimes a)^g = 1 \otimes a^g,$$

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where $m \in M$, $a \in A_n$. Construct the crossed product $\widetilde{\mathfrak{M}} = G \times \widetilde{M}$. Since $(1 \otimes P_0)\widetilde{\mathfrak{M}}(1 \otimes P_0) = M \otimes P_0$ is an approximately finite factor of type II_1 in the space $P_0\widetilde{H}$, it follows that $\widetilde{\mathfrak{M}}$ is also an approximately finite factor. (It is clear that the converse is true.)

Let N be the set of fixed elements of $\widetilde{M}$ with respect to the group G. Using the same device as in the proof of the lemma, we establish that N is a subfactor of $\widetilde{\mathfrak{M}}$. Since $\widetilde{\mathfrak{M}}$ is approximately finite, N is also approximately finite.

Note that N contains the factor $1 \times (M_0 \otimes 1)$, isomorphic to M_0, as a subfactor. We shall show that the approximate finiteness of N implies the approximate finiteness of M_0.

Denote the operators $1 \times (1 \otimes P_i)$ $(i = 0, 1, \dots, n-1)$ simply by P_i, and denote the commutative algebra which they generate, as before, by A_n. Next, denote the operators $[g \times (1 \otimes 1)]$ from $\widetilde{\mathfrak{M}}$ by U_g. Now consider the subalgebra $\widetilde{\mathfrak{M}}_n \subset \widetilde{\mathfrak{M}}$ generated by the operators P_i $(i = 0, 1, \dots, n-1)$ and U_g $(g \in G)$. It is clear that $\widetilde{\mathfrak{M}}_n$ is isomorphic to the full matrix algebra of order n. Then $\widetilde{\mathfrak{M}}_n$ contains a unitary operator U_h such that

$$U_h^* P_i U_h = P_{i+1} \quad \text{(indices are taken mod } n) \tag{1}$$

and $U_h^n = 1$. A more detailed analysis shows that U_h can be represented in the form (see (4))

$$U_h = \sum_{g \in G} P_g U_g, \tag{2}$$

where the P_g are mutually disjoint projections from A_n, and moreover

$$\sum_{g \in G} P_g = I, \qquad \sum_{g \in G} P_g^g = I. \tag{3}$$

It is clear that the operator U_h determines an outer automorphism h of the algebra A_n:

$$U_h^* a U_h = a^h \quad (a \in A_n).$$

Let us verify that U_h determines an outer automorphism of $1 \times \widetilde{M}$. Indeed, if $m \in 1 \times \widetilde{M}$, then

$$m^h = U_h^* m U_h = \sum_{g_1, g_2} U_{g_1}^* P_{g_1} m P_{g_2} U_{g_2} = \sum_{g \in G} U_g^* (m P_g) U_g = \sum_{g \in G} m^g P_g^g. \tag{4}$$

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But

$$\sum_{g\in G} m^g P_g^g \in 1 \times \widetilde{M}.$$

We shall show that $\widetilde{\mathfrak{M}}$ is the crossed product $G_h \times \widetilde{M}$, where G_h is the cyclic group of order n generated by U_h. First, $\widetilde{\mathfrak{M}}$ is generated by the operators from $1 \times \widetilde{M}$ and G_h. Second, the ope-

the operators U_h^i $(i = 0, 1, \dots, n-1)$ are orthogonal to the operators from $I \times \widetilde{M}$. Indeed, let $m \in I \times \widetilde{M}$ and let Tr be the trace in $\mathfrak{M}$; then

$$\mathrm{Tr}(mU_h) = \sum_{g\in G} \mathrm{Tr}(mP_g U_g) = 0,$$

since the U_g $(g \in G)$ are orthogonal to the operators from $I \times \widetilde{M}$ by the definition of the crossed product.

Denote by N_1 the subset of $\widetilde{M}$ invariant with respect to h. From (2) and (4) it follows that $N \subset N_1$. Since (2) is invertible, $N_1 \subset N$, and $N = N_1$.

Let us note that the factor N_1 is generated by elements of the form

$$\sum_{i=1}^{n-1} (mP_k)^{h^i}, \qquad m \in I \times (M \otimes I).$$

Define the mapping $\sigma : N \to N$

$$\sigma : \sum_{i=1}^{n-1} (mP_k)^{h^i} \to \sum_{i=1}^{n-1} (mP_{k+1})^{h^i} \quad \text{(the indices are taken modulo } n\text{)}.$$

An elementary verification shows that σ is an outer automorphism of N of order n. Denote by G_σ the group generated by σ. Since N is approximately finite, $G_\sigma \times N$ is also an approximately finite factor (Theorem A). Further, the set of elements of N invariant with respect to σ coincides with $I \times (M_0 \otimes I)$. From the lemma we conclude that $I \times (M_0 \otimes I)$, and consequently M_0, is an approximately finite factor. This completes the first part of the proof of Theorem 1.

The proof of the second part of the theorem now presents no difficulties.

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Note: Figure translations are in progress. See original paper for figures.

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