

# ON SOME PROPERTIES OF POSITIVE SOLUTIONS OF EVOLUTIONARY HYPOELLIPTIC EQUATIONS

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON SOME PROPERTIES OF POSITIVE SOLUTIONS OF EVOLUTIONARY HYPOELLIPTIC EQUATIONS

*(Presented by Academician I. G. Petrovskii on 10 VI 1968)*

Here several propositions will be established concerning the nature of positive solutions of certain evolutionary hypoelliptic equations. The means by which such propositions are proved are quite modest: the existence of a certain auxiliary function is used, as are interior estimates of solutions, estimates of solutions up to a piece of the initial hyperplane, and a certain invariance of the main characteristics of the equation under simple linear changes of the independent variables. It is precisely this that makes it possible to include in the investigation positive (bounded below) solutions of various classes of hypoelliptic equations with variable real coefficients. We shall state the results obtained for the case of an equation parabolic in the sense of Petrovskii of arbitrary order; possible generalizations will be indicated at the end of the paper.

**1. Basic lemmas.** Consider the equation

$$\mathcal{L}u \equiv -\frac{\partial u}{\partial t} + \sum_{|k| \leq 2m} a_k(t, x) D_x^k u = 0. \quad (1)$$

Assume that:

- 1)  $a_k(t, x)$  are real.
- 2) Equation (1) is uniformly parabolic (where it will be considered), i.e., there exists a positive constant  $\delta > 0$  such that for any real vector  $\sigma$

$$(-1)^m \sum_{|k|=2m} a_k(t, x) \sigma^k < -\delta |\sigma|^{2m}.$$

- 3)  $D_x^k a_k(t, x)$  are bounded functions satisfying a uniform Hölder condition (in the sense of the parabolic distance).

Let us note that under these conditions equation (1) has a fundamental solution  $\mathcal{L}(t, x, \tau, \xi)$ , satisfying, in the parametric variables  $\tau, \xi$ , the adjoint equation  $\mathcal{L}^*v = 0$ .

By  $\Pi_{(0,T)}^a$  we denote the parallelepiped  $0 < t \leq T$ ,  $|x_j| \leq a$ ,  $j = 1, 2, \dots, n$ . The following lemmas are basic.

**Lemma 1.** Let  $u(t, x)$  be a positive solution in  $\Pi_{(0,T)}^2$  of the equation  $\mathcal{L}u = f(t, x)$ , continuous in  $\Pi_{[0,T]}^2$ .

If

$$u|_{t=0, |x_j| \leq 2} < B,$$

$$\iint_{\Pi_{(0,T)}^2} |f(t, x)| dt dx = |f| < +\infty, \quad (2)$$

$$\iint_{\Pi_{(0,T)}^1} u(t, x) dt dx < 1,$$

then there exist positive  $a_1$ ,  $h_1 < T$  and  $\lambda_1 > 1$  such that

$$\iint_{\Pi_{(0, T-h_1)}^{1+a_1}} u(t, x) dt dx < \lambda_1(1 + B + |f|). \quad (3)$$

The constants  $a_1$ ,  $h_1$ ,  $\lambda_1$  depend only on the constant of parabolicity  $\delta$ , the maximum of  $D^k a_k(t, x)$ , and their Hölder constants.

**Lemma 2.** If all the assumptions formulated in Lemma 1 are satisfied, then for any  $h$ ,  $0 < h < h_1$  ( $h_1$  from Lemma 1), there exist positive  $a_2, \lambda_2$ , independent of  $h$ , such that

$$\iint_{\Pi_{(0, T-h)}^{1+a_2 h^{1/2m}}} u(t, x) dx dt < \lambda_2(1 + B + |f|). \quad (4)$$

As a useful consequence of Lemmas 1 and 2, let us note that if the condition of positivity of  $u(t, x)$  is replaced by the assumption that in  $\Pi_{(0,T)}^2$   $u(t, x)$  satisfies the inequality  $u(t, x) > -M$ , then for the function  $v(t, x) = u(t, x) + M$ , from inequality (2) there will follow the inequality

$$\iint_{\Pi_{(0, T-h)}^{1+a_2 h^{1/2}}} v(t, x) dt dx < \lambda_2(1 + B + M + |f|). \quad (5)$$

**Lemma 3.** If  $u(t, x)$  is a positive solution in  $\Pi_{(-T_2, T_2)}^2$  of the equation  $\mathcal{L}u = f$ , then for any positive  $h$ ,  $0 < h < h_1$ , from the fact that

$$\iint_{\Pi_{(-T_2, T_2)}^1} u(t, x) dt dx < 1,$$

there follows the inequality

$$\iint_{\Pi_{(-T_2+h, T_2-h)}^{1+a_2 h^{1/2m}}} u(t, x) dt dx < \lambda_2(1 + |f|). \quad (6)$$

**Lemma 4.** Let  $u(t, x)$  be a positive solution in  $\Pi_{(-T, 0)}^2$  of the equation  $\mathcal{L}u = f$ , satisfying the conditions

$$u|_{t=0, |x_j| \leq 2} < B_1,$$

$$\iint_{\Pi_{(-T+h_2, 0)}^1} u(t, x) dt dx < 1, \quad m = 2p + 1, \quad 0 < h_2 < T.$$

Then there exist positive constants  $a_3, \lambda_3$ , and  $h_3 < h_2$ , such that

$$\iint_{\Pi_{(-T+h_3, 0)}^{1+a_3}} u(t, x) dt dx < \lambda_3(1 + B_1 + |f|). \quad (7)$$

Lemmas 1, 3, and 4 are established by constructing an auxiliary function  $v(t, x)$  which vanishes, together with a sufficient number of derivatives, on the surface of a sphere and is such that in some spherical segment  $\mathcal{L}^*v \geq \eta_0 > 0$ . Then, using Green's formulas, one succeeds in estimating the integral over a part of the spherical segment (or half-segment) through the derivatives of the solution under study on the flat part of the boundary of the sector (in the case of a half-segment, the estimate includes the maximum of the original function on the additional flat part of the boundary). Then  $\Pi_{(0, T)}^1$  is covered from the sides (in Lemma 4, also from below) by a system of congruent spherical sectors whose flat parts of the boundaries lie strictly inside  $\Pi_{(0, T)}^1$ . Using the re-

using inequality (2) and interior estimates of solutions of the equation  $\mathcal{L}u = f$  (in the case of a half-sector, estimates are also needed up to a piece of the initial hyperplane), we arrive at the assertions of Lemmas 1, 3, and 4. For the proof of Lemma 2, one additionally uses the structure of the equation, which possesses a certain invariance of the principal characteristics under a weighted similarity transformation.

**2. On the growth character of solutions.** With the help of Lemmas 1 and 2 one establishes

**Theorem 1.** Let  $u(t, x)$  be a solution of the equation  $\mathcal{L}u = f$  in the layer  $\Pi_T$  ( $0 < t \leq T$ ,  $-\infty < x_j < +\infty$ ,  $j = 1, 2, \dots, n$ ), continuous for  $0 \leq t \leq T$ , and satisfying the conditions:

- 1)  $u(t, x) \geq -F_1(|x|)$ ;
- 2)  $u(0, x) \leq F_2(|x|)$ ;
- 3)  $\iint_{\Pi_{(0, T)}^{2, x}} |f(\tau, \xi)| d\tau d\xi \leq F_3(|x|)$ ,

where  $F_i(r)$ ,  $i = 1, 2, 3$ , are positive nondecreasing functions defined for  $0 \leq r < \infty$ ;  $\Pi_{(0, T)}^{a, x}$  is the parallelepiped  $0 < \tau < T$ ,  $|\xi_j - x_j| \leq a$ ,  $j = 1, 2, \dots, n$ .

Then

$$\iint_{\Pi_{(0, T-t+\eta)}^{1, x}} u(\tau, \xi) d\tau d\xi \leq C \exp \left[ c|x|^{\frac{2m}{2m-1}} \eta^{\frac{1}{1-2m}} \left( \sum_{j=1}^3 F_j(|x| + 2) + 1 \right) \right]. \quad (8)$$

Using Lemma 4 and the invariance of equation (1), in the case when it contains only the highest derivatives, under weighted similar dilations, one obtains

**Theorem 2.** Let  $u(t, x)$  be a positive solution, in the half-space  $t \leq 0$ , of the equation

$$\frac{\partial u}{\partial t} = \sum_{|k|=2m} a_k(t, x) D_x^k u,$$

which at  $t = 0$  grows no faster than a power. Then it has the same property throughout the entire half-space  $t \leq 0$ .

**3. On uniqueness of the solution of the Cauchy problem.** An immediate consequence of Theorem 1 is the following assertion.

**Theorem 3.** a) If  $u(t, x)$  is a solution in  $\Pi_T$  of the equation  $\mathcal{L}u = 0$ , satisfying the conditions

- 1)  $u|_{t=0} = 0$ ;
- 2)  $u(t, x) \geq -Ae^{|x|h(|x|)}$ ,

where  $h(r)$  is a positive nondecreasing function, then

$$u(t, x) \leq C \left[ (1 + A) \exp \left( c_1 |x|^{\frac{2m}{2m-1}} \right) + (|x| + 1)h(|x| + 1) \right].$$

b) If the conditions of part a) are fulfilled and

$$\int_1^\infty h(r)^{1-2m} dr = \infty,$$

then

$$u(t, x) \equiv 0 \quad \text{in } \Pi_T.$$

The second part of the theorem follows, in the case of equation (1), from assertion a) and results of S. Täcklind <sup>(4)</sup> and G. N. Zolotarev <sup>(5)</sup>.

The uniqueness theorem for a positive solution of the heat-conduction equation was established by Widder <sup>(3)</sup> (see also <sup>(2)</sup>).

#### 4. Some Generalizations

All the results formulated above extend, without changes in the proofs, to equations uniformly parabolic in the sense of Petrovskii of arbitrary order in  $t$ , to  $2\bar{b}$ -parabolic equations (1), and to equations involving products of such operators with different weights, as well as to equations obtained from those listed by replacing  $t$  by  $-t$  (backward-parabolic equations).

For a broad class of hypoelliptic equations, Lemmas 3 and 4 remain valid; their proof requires only the existence of interior estimates for solutions, as well as the uniqueness theorems for a positive solution of the Cauchy problem.

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