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Abstract

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MATHEMATICS

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EQUIVALENT REGULARIZATION OF BOUNDARY-VALUE PROBLEMS BY MEANS OF POTENTIALS

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1. One of the most interesting methods for solving boundary-value problems for partial differential equations is the regularization of these boundary-value problems by means of potentials. This method has repeatedly been applied by various authors to the solution of a number of concrete boundary-value problems —mainly the Dirichlet and Neumann problems for the Laplace and Helmholtz equations, for various problems of the theory of elasticity, and also for some boundary-value problems for polyharmonic equations. The corresponding works are well known, and there is no need to list them here. Further, methods of potential theory have been fruitfully used by a number of authors to obtain estimates for the solution of boundary-value problems near the boundary ([1,2]). In the paper of Ya. B. Lopatinskii ([9]) a way is indicated for finding potentials that reduce to regular integral equations any boundary-value problem for which this is possible in general. The condition formulated in ([9]) turned out, as was clarified later, to be necessary and sufficient for the normal solvability of a boundary-value problem in the sense of Noether.

However, the further development of potential theory encountered a number of difficulties, the chief of which are as follows: a) until recently the question of the existence of fundamental solutions in the large for elliptic equations was not clear; b) the question of the equivalence of the original boundary-value problem and the integral equations obtained under regularization had not been investigated in general form.

The question of fundamental solutions was essentially solved in the work of Yu. I. Lyubich ([3]) and in subsequent works on estimates for the Cauchy problem (see, for example, ([4])). In the present work the question of the equivalence of the original boundary-value problem and the integral equations obtained by regularization by methods of potential theory is partly solved. The basic idea is the introduction and special choice of a certain parameter. For two-dimensional boundary-value problems this method was set forth by the author in ([5]). In ([6]) the same method was used for some boundary-value problems of special form for the Helmholtz equation and the system of Maxwell equations. In the present work an arbitrary normally solvable boundary-value problem in n-dimensional space is considered. For simplicity the exposition is carried out for the example of a single equation, although the generalization to systems of equations presents no difficulty.

2. Potentials, as is known, provide a right regularizer of a boundary-value problem. An operator $B : X_2 \to X_1$ is called a right regularizer of an operator $A : X_1 \to X_2$ if $AB = I_2 + T$, where I_2 is the identity operator in X_2, and T is a completely continuous operator in X_2.

Theorem 1. *In order that the right regularizer B be an equivalent right regularizer for the operator A, it is necessary and sufficient that the dimension of the eigenspace of the operator B coincide with the number $(-\varkappa_A)$, where $\varkappa_A$ is the index of the operator A.*

It follows from this, in particular, that for the existence of an equivalent right regularizer it is necessary that the index of the original operator be nonpositive.

3. Let $\overline{\Omega}_0$ be some closed domain in R_n, whose boundary is an infinitely smooth compact manifold without boundary Γ_0. Let an elliptic differential operator $A(x, D)$ of order $2m$ be given in $\overline{\Omega}_0$. Here

$$x = (x_1, \dots, x_n), \quad D = (D_1, \dots, D_n), \quad D_j = -i\partial/\partial x_j.$$

We shall assume that the coefficients of the operator A are infinitely smooth functions in $\overline{\Omega}_0$.

Assumption 1. In the domain Ω_0, the equation $A(x, D)u = 0$ has a fundamental solution in the whole space $G(x, y)$, and near the singular point the equality

$$G(x, y) = |x - y|^{2m-n} \varphi(x, \omega, |x - y|) \tag{1}$$

holds (n odd, or n even but $2m - n < 0$), where φ is an infinitely smooth function of its arguments, $\omega = (x-y)/|x-y|$. In the case of even n, $2m-n \geq 0$, relation (1) is replaced by the following:

$$G(x, y) = |x - y|^{2m-n} \ln|x - y| \varphi(x, \omega, |x - y|). \tag{2}$$

Assumption 2. In the domain $\overline{\Omega}_0$, the Dirichlet problem for the equation $A(x, D)v = 0$ is uniquely solvable.

Obviously, if this assumption is satisfied, the Green' s function of the Dirichlet problem may be taken as the fundamental solution.

4. Let $\overline{\Omega}$ be some compact domain, $\overline{\Omega} \subset \Omega_0$. The boundary of $\overline{\Omega}$ is an infinitely smooth compact manifold without boundary Γ. Introduce the

potentials

$$w_k(x,\chi) = \int_\Gamma \chi(y) D_{\nu y}^{2m-k} G(x,y)\, d\Gamma_y, \qquad k = 1, 2, \dots, m, \tag{3}$$

where $D_\nu = -i\partial/\partial\nu$; $\partial/\partial\nu$ is the derivative in the direction normal to Γ. On the manifold Γ_0 these potentials satisfy the homogeneous Dirichlet conditions:

$$D_\nu^p w_k(x,\chi)\big|_{x\in\Gamma_0} = 0, \qquad p = 0, 1, \dots, m-1. \tag{4}$$

On the manifold Γ we obtain the relations

$$\lim_{x\to x_0\in\Gamma} D_\nu^p w_k(x,\chi) = \pm\varkappa_k^p(x_0, D)\chi + \lambda_0[\lambda_k^p(x_0, D)\chi], \tag{5}$$

where the plus sign refers to the limit from Ω, and the minus sign to the limit from $\Omega_0/\overline{\Omega}$. The symbols of the operators entering formula (5) will be written out below. Introduce the notation: $A_0(x,D)$ is the principal part of the operator A; $D = (D_1, \dots, D_n) = (D', D_n)$; $\xi = (\xi_1, \dots, \xi_n) = (\xi', \xi_n)$; $a_0(x_0)$ is the coefficient of D_n^{2m} in $A_0(x,D)$; $b_{p-k}(x_0,\xi')$ is the coefficient of z^{2m-1} in the remainder after division of z^{2m+p-k} by $A_0(x_0,\xi',z)$; for $p-k < -1$ we have $b_{k-p} \equiv 0$; $b_{-1} \equiv 1$; $Q_{p-k}(x_0,\xi',z)$ is a polynomial in (ξ',z), whose degree in z is no higher than $2m-2$; this polynomial is obtained by extracting from the remainder after division of z^{2m+p-k} by $A_0(x_0,\xi',z)$ the expression

$$\frac{1}{2m\, a_0(x_0)} A'_{0z};$$

for $p-k < -1$ we have

$$Q_{p-k} \equiv z^{2m+p-k}.$$

Let the equation $A(x,D)u = 0$ be written in such a coordinate system that the origin coincides with the point $x_0 \in \Gamma$, and the axis ox_n is directed along the normal to Γ at the point x_0. Then, for the symbols of the operators entering (5), we obtain the expressions

$$\tilde\lambda_0(\xi') = -\frac{(2m-2)!}{2^{2m-1}[(m-1)!]^2}\,\frac{1}{|\xi'|^{2m-1}}; \tag{6}$$

$$\tilde\varkappa_k^p(x_0,\xi') = \frac{(-1)^{k-1} b_{p-k}(x_0,\xi')}{2a_0(x_0)}\, i; \tag{7}$$

$$\tilde\lambda_k^p(x,\xi') = (-1)^k \frac{2^{2m-1}[(m-1)!]^2}{2\pi(2m-2)!} |\xi'|^{2m-1} \int_+ \frac{Q_{p-k}(x_0,\xi',z)}{A_0(x_0,\xi',z)}\, dz. \tag{8}$$

The integral is taken over a contour enclosing the z-zeros of the function $A_0(x_0,\xi',z)$ in the upper half-plane. The operators $\mathfrak{x}_k^p$ have orders $p-k+1$, with $\mathfrak{x}_k^p = 0$ for $p-k+1 < 0$. The operators λ_k^p have orders $2m+p-k \geqslant 0$.

5. For the potentials w_k the inequalities hold

$$\|w_k(x,\chi)\|_{H_l(\overline{\Omega})} \leqslant B\|\chi\|_{H_{l-k+1/2}(\Gamma)}, \qquad l \geqslant k; \tag{9}$$

$$\|w_k(x,\chi)\|_{H_l(\overline{\Omega})} \leqslant B\|\chi\|_{H_{1/2}(\Gamma)}, \qquad l \geqslant k; \tag{10}$$

$$\|w_k(x,\chi)\|_{C_{l,\alpha}(\overline{\Omega})} \leqslant B\|\chi\|_{C_{l-k+1,\alpha}(\Gamma)}, \qquad l \geqslant k-1, \quad 0<\alpha<1; \tag{11}$$

$$\|w_k(x,\chi)\|_{C_{l,\alpha}(\overline{\Omega})} \leqslant B\|\chi\|_{C_{0,\alpha}(\Gamma)}, \qquad l < k-1. \tag{12}$$

Inequalities (9) and (10) are easy to prove using a priori estimates for the solution of the Dirichlet problem and formula (5). Inequalities (11) and (12) are proved by the methods presented in ([2],[7]).

6. Consider in the domain Ω the boundary-value problem

$$A(x,D)u = 0 \quad \text{in } \Omega; \tag{13}$$

$$B_j(x_0,D)u \equiv \lim_{x\to x_0\in\Gamma} \sum_{p=0}^{n_j} B_{jp}(x_0,D')D_\nu^p u(x) = f_j(x_0); \tag{14}$$

$$j = 1,2,\dots,m.$$

Here D' is the collection of derivatives in the tangential directions.

We shall seek the solution of the posed problem in the form

$$u(x) = \sum_{k=1}^{2m} w_k(x,\chi_k). \tag{15}$$

Substituting (15) into the boundary conditions and using (5), we obtain for the densities χ_k a system of integro-differential equations. Subjecting χ_k to the additional conditions

$$\sum_{k=1}^{2m} C_{jk}(x_0,D')\chi_k = \overline{\chi}_j; \qquad \sum_{k=1}^{2m} \mathfrak{D}_{jk}(x_0,D')\chi_k = 0, \quad j=1,2,\dots,m, \tag{16}$$

where

$$C_{jk}(x_0, D') = \sum_{p=0}^{n_j} B_{jp}^0(x_0, D')\, \mathfrak{x}_k^{0p}(x_0, D'),$$

$$\mathfrak{D}_{jk}(x_0, D') = \sum_{p=0}^{n_j} B_{jp}^0(x_0, D')\, \lambda_k^{0p}(x_0, D')$$

(the zero above denotes the principal part of the corresponding operator), q is an arbitrary parameter, and $\overline{\chi}_j$ are new unknown functions, we obtain for $\overline{\chi}_j$ a regular system of integro-differential equations

$$\overline{\chi}_j(x_0) + \sum_{p=1}^{m} T_{jp}(x_0, q)\overline{\chi}_p = f_j(x_0), \quad j = 1, 2, \dots, m. \tag{17}$$

Assumption 3. The boundary-value problem under consideration is normally solvable, i.e. it satisfies the Shapiro–Lopatinskii condition ([8],[9]). This is equivalent to the ellipticity of system (16) (in the sense of Douglis–Nirenberg).

There is no need to solve system (16) exactly with respect to χ_k; it is sufficient to construct a regularizer, which, under Assumption 3, certainly exists.

7. Put

$$B\chi \equiv \sum_{p=1}^{2n} w_k(x, \chi_k) = v(x). \tag{18}$$

According to Theorem 1, for the equivalence of system (17) to the original boundary-value problem it is necessary and sufficient that the equation $v(x) = 0$ in Ω have exactly $(-\varkappa_A)$ independent solutions, where $\varkappa_A$ is the index of the boundary-value problem under consideration, which is assumed to be nonpositive. At the same time one should not forget that the densities χ_k must satisfy system (16), at least up to operators of lower order.

Introduce the operators:

$$\overset{0}{\delta}{}^{(0)} \equiv 1,$$

$$\overset{0}{\delta}{}^{(p)}(x, 0) = (-1)^p \left[D_\nu^p + 2ia_0(x_0) \sum_{k=1}^{p} \overset{0}{\varkappa}{}_k^p(x_0, D')\, \overset{0}{\delta}{}^{(k-1)}(x_0, D) \right], \quad p > 1. \tag{19}$$

Using the jumps of the potentials in passing through Γ and system (16), we obtain the following relations (under the assumption that system (16) has been solved exactly):

$$\left\{ \sum_{k=1}^{2m} \mathfrak{D}_{jk}(x_0, D') \overset{0}{\delta}{}^{(k-1)} v - q^{2m-1} \sum_{k=1}^{2m} C_{jk}(x_0, D') \overset{0}{\delta}{}^{(k-1)}(x_0, D')\, v + \dots \right\}_{x_0 \in \Gamma} = 0, \quad j = 1, 2, \dots, m. \tag{20}$$

The dots denote derivatives of lower order. By arbitrary functions v on Γ we understand here the limit from the domain $\Omega_0 \setminus \overline{\Omega}$. Adding to conditions (20) the equation

$$A(x, D)v = 0 \quad \text{in } \Omega_0 \setminus \overline{\Omega} \tag{21}$$

and the homogeneous Dirichlet conditions

$$D_\nu^p v\big|_{\Gamma_0} = 0, \quad p = 0, 1, \dots, m-1, \tag{22}$$

we obtain a boundary-value problem, which we call auxiliary.

Equivalence Theorem. *In order that system (17) be equivalent to the original boundary-value problem, it is sufficient that the auxiliary problem (20)–(22) have exactly* $(-\varkappa_A)$ *independent solutions for at least one value of the parameter* q*. It is precisely for this* q *that equivalence will be obtained.*

In the case $\varkappa_A = 0$, the auxiliary problem is investigated for uniqueness. In this case one may use the results of work ([10]), which is a certain development of the ideas of M. S. Agranovich and M. I. Vishik ([11]). If system (16) is not solved exactly, but a regularizer is constructed, then the boundary condition (20) is modified. Of interest is the case where system (20) can be transformed into a semi-bounded one (see ([11])) with the same leading terms with respect to derivatives.

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Note: Figure translations are in progress. See original paper for figures.