ON THE INDEX OF FAMILIES OF PSEUDODIFFERENTIAL OPERATORS ON MANIFOLDS WITH BOUNDARY

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Abstract

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ON THE INDEX OF FAMILIES OF PSEUDODIFFERENTIAL OPERATORS ON MANIFOLDS WITH BOUNDARY

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In the present paper a homotopic method is indicated for computing the index of a locally trivial family of Fredholm structures of pseudodifferential operators on manifolds with boundary, studied in ([4]), and in particular of families of elliptic differential boundary-value problems. This clarifies the homotopic structure of the space of Fredholm structures of pseudodifferential operators (cf. the program of I. M. Gel' fand ([7])).

Let us note that the index of a family of pseudodifferential operators on closed manifolds was computed in the work of B. Shih ([3]).

1. Let $M \to X$ be a locally trivial fibration over a Hausdorff compactum X, whose typical fiber is a smooth compact Riemannian manifold with boundary (smoothness everywhere means belonging to the class $\mathcal{C}^\infty$). Associated with this fibration are: the fibration $M' \to X$ of the boundaries M'_x of the fibers M_x of the fibration M (here and below $x \in X$); the fibration $T \to X$ of the tangent fibrations T_x of the manifolds with boundary M_x; the fibration $N \to X$ of the normal fibrations N_x of the boundaries M'_x; the fibration $T' \to X$ of the tangent fibrations T'_x of the boundaries M'_x.

Let $E \to X$ denote the fibration of Hermitian fibrations E_x over M_x, associated with the fibration $M \to X$. This fibration generates over X the Banach fibrations $\mathcal{H}_0^{s,p}(M;E)$, $\mathcal{H}^{s,p}(M;E)$, $\mathcal{B}^{s,p}(M';E|M')$, $-\infty < s < \infty$, $1 < p < \infty$, whose fibers are the Banach spaces $\mathcal{H}_0^{s,p}(M_x;E_x)$, $\mathcal{H}^{s,p}(M_x;E_x)$, $\mathcal{B}^{s,p}(M'_x;E|M'_x)$ ([4]). Here and below a vertical bar denotes restriction.

For $ps > 1$ the restriction epimorphism is defined:

$$\delta : \mathcal{H}^{s,p}(M;E) \to \mathcal{B}^{s-1/p,p}(M';E|M').$$

For $ps < p-1$ the dual cosection homomorphism is defined

$$\delta' : \mathcal{B}^{s+1-1/p,p}(M';E|M') \to \mathcal{H}_0^{s,p}(M;E).$$

Let $M_1 = M \setminus M'$. A homomorphism of fibrations

$$P : \mathcal{C}_0^\infty(M_1; E_1) \to \mathcal{C}^\infty(M_1; E_2)$$

will be called **pseudodifferential** of class $\mathcal{P}^r(M; E_1, E_2)$, if P_x are pseudodifferential operators of classes $\mathcal{P}^r(M_x; E_{1x}, E_{2x})$. These homomorphisms extend (see (^{4})) to homomorphisms

$$P_{(0)}^{(s)} : \mathcal{H}_0^{s,p}(M; E_1) \to \mathcal{H}^{s-r,p}(M; E_2).$$

The family of symbols $\sigma(P_x)$ of the operators P_x determines a homomorphism of fibrations $\bar{\sigma}(P) : E_1^\pi \to E_2^\pi$, where E_1^π, E_2^π are the lifts of the fibrations E_1, E_2 to the space T_0 of nonzero elements of the fibration T. Similarly, the family of indicators $\iota(P_x)$ of the pseudodifferential operators P_x (^{4}) determines a homomorphism $\iota(P)$ of the corresponding Banach fibrations over T_0', the space of nonzero elements from T'.

For what follows fix a number p strictly between 1 and ∞. Let $P \in \mathcal{P}^r(M; E_1, E_2)$, $P_1 \in \mathcal{P}^{r_1}(M; F_1, E_2)$, $P_2 \in \mathcal{P}^{r_2}(M; E_1, F_2)$, where F_1, F_2 are fibrations over X of Hermitian fibrations F_{1x}, F_{2x} over M_x. For any s such that $s_1 = s - r + r_1 < 1 - 1/p$, $s_2 = s - r_2 > 1/p$,

define an augmentation of the pseudodifferential homomorphism

$$\mathfrak{P}_{(0)}^s : \mathcal{H}_0^{s,p}(M; E_1) \times \mathcal{B}^{s_1+1-1/p,p}(M'; F_1|M') \to$$

$$\to \mathcal{H}^{s,p}(M; E_2) \times \mathcal{B}^{s_2-1/p,p}(M'; F_2|M')$$

of the form

$$\mathfrak{P}_{(0)}^s(u, v) = (P_{(0)}^s u + P_{1(0)}^{s_1} \delta^* v,\ \delta P_{2(0)}^{s_2} u).$$

The augmentation $P_{(0)}^s$ is called Fredholm if all the corresponding augmentations $\mathfrak{P}_{(0)x}^s$ are Fredholm (4). For Fredholm augmentations the index is defined in the sense of (5).

2. A pseudodifferential homomorphism $P \in \mathcal{P}^r(M; E_1, E_2)$ will be called simple if the restriction of the homomorphism $\bar{\sigma}(P)$ to $T|M'$ has the form $\bar{\sigma}(P)(\xi) = (\nu_\xi + i|\tau_\xi|)^r \beta^\pi$, where $\xi \in T|M'$, (ν_ξ, τ_ξ) are the (scalar) normal and tangential components of the vector ξ, and β^π is the lifting of some isomorphism $\beta \in \mathrm{Iso}(E_1|M', E_2|M')$.

Let $U \to X$ be a fibration of tubular neighborhoods U_x of the boundaries M_x' in M_x. Let φ be a continuous function on U such that $\varphi|U_x \in \mathcal{E}^\infty(U_x)$, $0 \le \varphi \le 1$, $\varphi|M' = 1$. Introduce pseudodifferential homomorphisms $P_t^r(E) \in \mathcal{P}^r(M; E, E)$ with real t, for which

$$\bar{\sigma}(P_t^r(E))(\xi) = \left[(\nu_\xi - i|\tau_\xi|)^t(\nu_\xi + i|\tau_\xi|)^{r-t}\varphi(m_\xi) + i|\xi|^r(1-\varphi(m_\xi))\right] 1_E,$$

where $\xi \in T_0$, m_ξ is the projection of ξ onto M, (ν_ξ, τ_ξ) are the (scalar) normal and tangential components of ξ for $m_\xi \in U$, and 1_E is the identity endomorphism in E.

Define: the augmentation $[\mathfrak{P}^r_{r-s}(E)]^s_{(0)}$ by means of $P_1 = 0,\ P_2 = 0$; the augmentation $[\mathfrak{P}^r_{r-s-1}(E)]^s_{(0)}$ by means of $P_1 = 0,\ P_2 = P^r_{r-s}(E)$; the augmentation $[\mathfrak{P}^r_{r-s+1}(E)]^s_{(0)}$ by means of $P_1 = P^r_{r-s}(E),\ P_2 = 0$.

Lemma 1. The augmentations $[\mathfrak{P}^r_{r-s}(E)]^s_{(0)}$, $[\mathfrak{P}^r_{r-s-1}(E)]^s_{(0)}$, $\mathfrak{P}^r_{r-s+1}(E)]^s_{(0)}$ are Fredholm. Their indices are equal to $0 \in K(X)$.

Theorem 1. Let $\mathfrak{P}^s_{(0)}$ be a Fredholm augmentation of a homomorphism $P \in \mathscr{P}^r(M; E_1, E_2)$. Then there exist such fibrations $\widetilde{E} = E_1 \oplus E$, $\widetilde{F}_1 = F_1 \oplus F$, $\widetilde{F}_2 = F_2 \oplus F$, that the Fredholm augmentation

$$\mathfrak{P}^s_{(0)} \oplus [\mathfrak{P}^r_{r-s}(E)]^s_{(0)} \oplus [\mathfrak{P}^r_{r-s+1}(\widetilde{F}_2)] \oplus [\mathfrak{P}^r_{r-s-1}(\widetilde{F}_1)]^s_{(0)}$$

is homotopic in the space of Fredholm augmentations to the Fredholm augmentation $\widetilde{\mathfrak{P}}^s_{(0)}$ of some simple pseudodifferential operator $\widetilde{P}$.

The indicated homotopy is constructed effectively, on the basis of linearization (cf. (1)).

Remark. From the theorem there follows, in particular, the homotopic triviality of the restriction of the homomorphism $\sigma(P)$ to every tangent sphere of the manifolds with boundary M_x. In the case where P is a differential operator admitting an elliptic boundary-value problem, this was proved by M. Atiyah and R. Bott in (1). Theorem 1 may be regarded not only as a generalization, but also as a refinement of the result of M. Atiyah and R. Bott in view of the special character of the homotopy.

Theorem 2. The index of the augmentation $\mathfrak{P}^s_{(0)}$ is equal to the index of the augmentation $\widetilde{\mathfrak{P}}^s_{(0)}$.

3. M. Atiyah and I. Singer computed the index of a Fredholm pseudodifferential homomorphism for empty M' and one-point X. One of their methods (6) directly generalizes to the computation of the index in the sense of (5) of a Fredholm simple pseudodifferential homomorphism. For this purpose consider an embedding i of the fibration M into a trivial finite-dimensional vector fibration V over X. Following (6), introduce an almost complex structure in the fiberwise tangent fibration TW of a normal tubular neighborhood W of the space M in V. Then the Thom isomorphism $\varphi_i : K(T) \to K(TW)$ is defined (6). The embedding $k : W \to V$ generates the imbedding

embedding $k_! : K(TW) \to K(TV)$. On the other hand, if the zero section $j : X \to V$ of the bundle V is regarded as an embedding in V of the identity family of zero-dimensional manifolds $X \to X$, then there is a Thom isomorphism $\varphi_j : K(X) \to K(TV)$.

If P is an elliptic simple homomorphism, then the homomorphism $\bar{\sigma}(P)$ determines a certain element $[\bar{\sigma}(P)] \in K(T)$ ([6]).

Theorem 3. Let $P \in P^r(M; E_1, E_2)$ be a Fredholm simple pseudodifferential homomorphism. Then the index of the homomorphism P_0^s is equal to

$$\varphi_j^{-1} k_! \varphi_i [\bar{\sigma}(P)].$$

Let $F_1^{\pi'}, F_2^{\pi'}$ be the lifts of the bundles $F_1|M'$, $F_2|M'$ to T_0'. If P_0^s is a Fredholm framing of a simple morphism P by means of P_1, P_2, then the product of homomorphisms

$$\delta^s_{l(0)}(P_2)\, t^s_{(0)}(P)^{-1}\, t^{s_1}_{(0)}(P_1)\delta^*$$

defines an isomorphism $F_1^{\pi'} \to F_2^{\pi'}$, which may be regarded as the symbol of a certain Fredholm homomorphism

$$P' \in P^{r_1+r_2-r}(M'; F_1|M', F_2|M').$$

Theorem 4. If $\mathfrak{P}^s_{(0)}$ is a Fredholm framing of a simple homomorphism P, then the index $\operatorname{ind} \mathfrak{P}^s_{(0)}$ is equal to the sum of the indices

$$\operatorname{ind} P^s_{(0)} + \operatorname{ind} {P'}^s_{(0)}.$$

Theorems 1-4 give a homotopy method for computing the index of any Fredholm framing.

4. We shall say that a homomorphism $P \in \mathcal{P}^r(M; E_1, E_2)$ satisfies condition $(\mathcal{D})$ if all operators P_x satisfy condition $(\mathcal{D})$ from ([4]). In this case a $(\mathcal{D})$-homomorphism is defined

$$P^s : \mathcal{H}^{s,p}(M; E_1) \to \mathcal{H}^{s,p}(M; E_2).$$

Let $P_1 \in P^{r_1}(M; F_1, E_1)$, $P_2 \in \mathcal{P}^{r_2}(M; E_1, F_2)$. If P and P_2 satisfy condition $(\mathcal{D})$, then one can define a $(\mathcal{D})$-framing

$$\begin{aligned}\mathfrak{P}^s : \mathcal{H}^{s,p}(M; E_1) \times \mathcal{B}^{s+1-1/p,p}(M'; F_1|M') \to \\ \to \mathcal{H}^{s-r,p}(M; E_2) \times \mathcal{B}^{s_2-1/p,p}(M'; F_2|M')\end{aligned}$$

of the form

$$\mathfrak{P}^s(u, v) = (P^{su} + P^{s_1}_{1,0}\delta^* v,\ \delta P_2^{s_2} u),$$

under the conditions $s_1 = s - r + r_1 < 1 - 1/p$, $s_2 = s - r_2 > 1/p$.

Lemma 2. If $P \in \mathcal{P}^r(M; E_1, E_2)$ satisfies condition $(\mathcal{D})$, then the product of homomorphisms $PP_r^0(E_1)$ belongs to $\mathcal{P}^r(M; E_1, E_2)$. (Here $P_r^0(E_1)$ is the homomorphism from item 2.)

Therefore to a $(\mathcal{D})$-framing $\mathfrak{P}^s$ one can assign the framing $(\mathfrak{P}_0)^s_{(0)}$ of the homomorphism

$$P_0 = PP_s^0(E_1)$$

by means of

$$P_{10} = P_1, \qquad P_{20} = P_2 P_s^0(E_1).$$

Since the homomorphism $[P_s^0(E_1)]_0^s$ is Fredholm and its index is equal to 0 (by Lemma 1), it follows that

Theorem 5. A $(\mathcal{D})$-framing $\mathfrak{P}^s$ is Fredholm if and only if the framing $(\mathfrak{P}_0)^s_{(0)}$ is Fredholm. Moreover, the indices of the two framings coincide.

Remark. Theorem 5 indicates, in particular, a method for computing the index of an elliptic differential boundary-value problem, which gives a solution of I. M. Gel' fand' s problem ([7]) (cf. ([1])).

5. The results set forth generalize to Fredholm framings of systems of A. Douglis–L. Nirenberg pseudodifferential homomorphisms.

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Note: Figure translations are in progress. See original paper for figures.