SELF-ADJOINT EXTENSIONS OF DIFFERENTIAL OPERATORS IN A SPACE OF VECTOR FUNCTIONS

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Abstract

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SELF-ADJOINT EXTENSIONS OF DIFFERENTIAL OPERATORS IN A SPACE OF VECTOR FUNCTIONS

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1. In the present note we establish the general form of self-adjoint boundary-value problems on the finite interval $[0, b]$ for differential equations $l[y] = \lambda y$ of arbitrary order m with continuous operator coefficients. For a scalar quasi-differential expression of even order with real coefficients, a description of all self-adjoint extensions on the interval $[0, b]$ was given in the work of M. G. Krein ([1]) and is presented in the monographs ([2],[3]). For the algebraic study of scalar boundary-value problems (Bôcher, etc.) see ([4]).
2. The basis of the proposed investigation is the concept, introduced by us, of a Hermitian relation.

Definition. A binary relation θ given in some Hilbert space H is called Hermitian if from $x\theta x'$, $y\theta y'$, where $x, x', y, y' \in H$, it follows that

$$(x', y) - (x, y') = 0, \tag{1}$$

and from the validity of (1) for certain $x, x' \in H$ with all pairs $y\theta y'$ it follows that also $x\theta x'$.

For each pair $x\theta x'$ construct the vectors $x^{\pm} = x' \pm ix$. By the equality $U_\theta x^+ = x^-$ a unitary operator U_θ is defined, which we shall call the Cayley transform of the Hermitian relation θ.

Theorem 1. Whatever the self-adjoint operator A and the unitary operator U, the relation defined by either of the equations

$$\cos A \cdot x' - \sin A \cdot x = 0, \tag{2}$$

$$(U - I)x' + i(U + I)x = 0, \tag{3}$$

is Hermitian. Conversely, every Hermitian relation $x\theta x'$ is representable in the forms (2) and (3), where the unitary operators $-e^{2iA}$ and U determine the relation θ uniquely and are its Cayley transform U_θ.

Corollary. Any Hermitian relation $x\theta x'$ can be represented in the form $x' = A_1x + x^\perp$, where $x \in H_1$, H_1 is some subspace of H, $x^\perp \in H_1^\perp$, and A_1 is a self-adjoint operator in H_1, possibly unbounded*.

Theorem 2. Let B and C be arbitrary bounded operators on all of H. The relation θ defined by them,

$$x\theta x' \leftrightarrow Cx' - Bx = 0 \tag{4}$$

is Hermitian if and only if the operators $B \pm iC$ are invertible on their ranges and the operator $U = (B+iC)^{-1}(B-iC)$ is unitary. Under these conditions the relation θ can be represented through the operator U by formula (3).

* Thus, the known general form of self-adjoint boundary conditions for elliptic differential equations ([5]) also admits interpretation from the point of view of the concept of Hermitian relations.

Corollary. If $\dim H < \infty$, relation (4) is Hermitian if and only if $BC^* = CB^*$ and $\det(BB^* + CC^*) \neq 0$.

We shall say that a relation θ is the closure of θ if the graph $\tilde{\theta}$ in $H \oplus H$ is the closure of the graph of θ.

Theorem 3. Let the operators B and C in (4) be arbitrary (possibly unbounded, nonclosed, and with domains D_B and D_C not dense in H). In order that the closure $\tilde{\theta}$ of the relation $\theta(4)$ be Hermitian, the following conditions are necessary: 1. The linear span $D_B \cup D_C$ is dense in H. 2. The operators $B \pm iC$ are invertible on their ranges. 3. The operator

$$U_1 = (B+iC)^{-1}(B-iC)$$

in $H_1 = D_B \cap D_C$ has a closure $\overline{U}_1$, which is unitary. 4. $B\{D_B \cap D_C^\perp\} = \{0\}$, $C\{D_C \cap D_B^\perp\} = \{0\}$. Conditions 1-4, together with condition 5.

$$\overline{D_B \cap D_C} = \overline{D_B} \cap \overline{D_C},$$

become sufficient for the Hermiticity of $\tilde{\theta}$, and the Cayley transform of the relation $\tilde{\theta}$ is then given by the formula

$$U_{\tilde{\theta}} = \overline{U}_1 \oplus I_{D_B^\perp} \oplus (-I_{D_C^\perp}).$$

3. Let $\mathscr{H}(0,b)$ be the Hilbert space of vector-functions with values in the separable Hilbert space H and with scalar product

$$\langle x, y\rangle = \int_0^b (x(t), y(t))_H \, dt.$$

Consider in $\mathcal{H}(0,b)$ a differential operation $l[y]$ of order m. For $m = 2n$ put

$$l[y] = \sum_{k=1}^{n} (-1)^k \{(p_{n-k} y^{(k)})^{(k)} - i[(q_{n-k} y^{(k)})^{(k-1)} + (q_{n-k} y^{(k-1)})^{(k)}]\} + p_{ny}. \quad (5)$$

Let all operator coefficients be self-adjoint:

$$p_k(t) = p_k^*(t), \qquad q_k(t) = q_k^*(t) \quad (6)$$

and depend continuously on t, together with their derivatives up to order $n-k$ inclusive, and suppose that $p_0^{-1}(t)$ exists and is bounded for $t \in [0,b]$*. Define for the operation (5) the quasiderivatives $y^{[k]}$ by the formulas**

$$y^{[j]} = y^{(j)} \quad (j = 0, 1, \dots, n-1); \qquad y^{[n]} = p_0 y^{(n)} - i q_0 y^{(n-1)},$$

$$y^{[n+k]} = -\frac{d}{dt} y^{[n+k-1]} + p_{ky}^{(n-k)} + i\,[q_{k-1} y^{(n-k+1)} - q_{ky}^{(n-k-1)}]$$

$$(k = 1, \dots, n; \quad q_n \equiv 0; \quad l[y] \equiv y^{[2n]}).$$

Let L be the operator generated by the expression $l[y]$ on the set D of all such $y(t) \in \mathcal{H}(0,b)$ with $m-1$ absolutely continuous (quasi-)derivatives for which $l[y] \in \mathcal{H}(0,b)$, and let L_0 be the restriction of L defined by the conditions

$$y_0 = y_0^{[1]} = \dots = y_0^{[m-1]} = 0, \qquad y_b = y_b^{[1]} = \dots = y_b^{[m-1]} = 0, \quad (7)$$

where $y_0^{[k]} = y^{[k]}(0)$, $y_b^{[k]} = y^{[k]}(b)$, $k = 0, 1, \dots$.

Denote

$$H^m = H \oplus \dots \oplus H$$

(m summands), and let $(\cdot,\cdot)_m$ be the scalar product in H^m. To each vector-function $u(t) \in D$ we associate a pair of vectors $\hat{u}, \hat{u}' \in H^m$ (for $m = 2n$):

$$\hat{u} = \{u_0, u_0', \dots, u_0^{(n-1)}, u_b, u_b', \dots, u_b^{(n-1)}\},$$

$$\hat{u}' = \{u_0^{[2n-1]}, u_0^{[2n-2]}, \dots, u_0^{[n]}, -u_b^{[2n-1]}, -u_b^{[2n-2]}, \dots, -u_b^{[n]}\}. \quad (8)$$

* These requirements are easily weakened by considering l as a quasidifferential operation.

** For $q_0 = q_1 = \dots = q_{n-1} = 0$ our definition coincides with that adopted in ([1–3]).

Then for operation (5) Lagrange' s identity is written in the form

$$\langle l[x], y\rangle - \langle x, l[y]\rangle = (\hat{x}', \hat{y})_m - (\hat{x}, \hat{y}')_m. \quad (9)$$

Lemma 1. *The equality $L_0^* = L$ is valid.*

An essential point in the proof of Lemma 1 is

Lemma 2. *The manifold of vector solutions $y = y(t)$ of the homogeneous equation of arbitrary order m:*

$$y^{(m)} + g_1(t)y^{(m-1)} + \cdots + g_m(t)y = 0$$

with arbitrary continuous operator coefficients $g_k(t)$ forms a subspace in $\mathcal{H}(0,b)$ (i.e., a closed one), since the dependence between the solutions $y(t) \in \mathcal{H}(0,b)$ and their Cauchy data is mutually continuous.

From (9), Theorem 1, and Lemma 1 there follows

Theorem 4. *Between the self-adjoint extensions $\widetilde{L}$ of the operator L_0 (5), (7) and the Hermitian relations θ in H^{2n} there exists a one-to-one correspondence, by virtue of which every $\widetilde{L}$ is generated by the operation $l[y]$ (5) and boundary conditions of any of the forms*

$$\cos \hat{A} \cdot \hat{y}' - \sin \hat{A} \cdot \hat{y} = 0, \tag{10}$$

$$(\hat{U} - \hat{I})\hat{y}' + i(\hat{U} + \hat{I})\hat{y} = 0, \tag{11}$$

where $\hat{A}, \hat{U}$ are respectively a self-adjoint and a unitary operator in H^{2n}, and $\hat{y}, \hat{y}' \in H^{2n}$ are determined from $y(t) \in D$ by formula (8). Conversely, any of these boundary conditions determines some self-adjoint extension $\widetilde{L}$ of the operator L_0.

Example 1. Separated self-adjoint boundary conditions for $t = 0$ are always reduced to the form*

$$\cos \hat{A}_0 \cdot \hat{y}_0' - \sin \hat{A}_0 \cdot \hat{y}_0 = 0, \tag{12}$$

where $\hat{A}_0$ is an arbitrary self-adjoint operator in H^n,

$$\hat{y}_0 = \{y_0, y_0, \dots, y_0^{(n-1)}\}, \qquad \hat{y}_0' = \{y_0^{[2n-1]}, y_0^{[2n-2]}, \dots, y_0^{[n]}\}.$$

Example 2. Generalized periodic conditions:

$$y_b^{[2n-k]} = U_k y_0^{[2n-k]}, \qquad y_b^{(k-1)} = U_k y_0^{(k-1)}, \quad k = 1, \dots, n;$$

U_k are unitary operators in H.

4. Let now l be an operation of odd order $m = 2n + 1$

$$l[y] = \sum_{k=0}^{n} (-1)^k \left\{ i \left[(q_{n-k} y^{(k)})^{(k+1)} + (q_{n-k} y^{(k+1)})^{(k)} \right] + (p_{n-k} y^{(k)})^{(k)} \right\}, \tag{13}$$

where $q_0^{-1}(t)$ exists and is continuous for $t \in [0, b]$, and (6) is satisfied. For operation (13) we define quasiderivatives

$$y^{[j]} = y^{(j)} \quad (j = 0, 1, \dots, n-1), \qquad y^{[n]} = -iq_0 y^{(n)},$$

$$y^{[n+k+1]} = -\frac{d}{dt} y^{[n+k]} + p_k y^{(n-k)} + i\left[q_k y^{(n-k+1)} - q_{k+1} y^{(n-k-1)}\right]$$

$$(k = 0, 1, \dots, n;\ q_{n+1} \equiv 0;\ l[y] \equiv y^{[2n+1]}).$$

Let $H_t^{\pm}$ be invariant subspaces of the operator $q_0(t)$ such that $q_0(t) > 0$ on H_t^+ and $q_0(t) < 0$ on H_t^-, and let $P_t^{\pm}$ be the orthoprojectors onto $H_t^{\pm}$, respectively. Put

$$q_{\pm}(t) = \pm q_0(t) P_t^{\pm}, \qquad Q_t^{\pm} = q_+^{1/2}(t) \pm q_-^{1/2}(t).$$

From the properties of $q_0(t)$ it follows that $\dim H_b^{\pm} = \dim H_0^{\pm}$. Let U_q be an arbitrary but fixed unitary operator in H carrying $H_b^{\pm}$ into $H_0^{\pm}$. To each $v(t) \in D$ (see item 3 for $m = 2n+1$) we associate a pair $\hat{v}, \hat{v}' \in H^{2n+1}$:

$$\hat{v} = \{Q_0^+ v_0^{(n)} + U_{qQ} b^+ v_b^{(n)},\ v_0,\ v_0', \dots, v_0^{(n-1)},\ v_b,\ v_b', \dots, v_b^{(n-1)}\}, \tag{14}$$

* For a scalar real equation of even order, conditions of the form (12) are given as self-adjoint in [6]. For Schr

$$\hat{v}' = \{iU_{qQ_{bv}} b^{(n)} - iQ_0^- v_0^{(n)},\ v_0^{[2n]},\ v_0^{[2n-1]}, \dots,\ v_0^{[n+1]},\ -v_b^{[2n]},\ -v_b^{[2n-1]}, \dots, \dots,\ -v_b^{[n+1]}\}. \tag{14}$$

Lemma 3. For the operator l (13), Lagrange' s identity has the form (9), where $\hat{x}, \hat{y}, \hat{x}', \hat{y}'$ are defined by formulas (14).

Theorem 5. For the operator L_0 (13), (7) of odd order, Theorem 4 is valid with the replacement in its formulation of $l[y]$ (5) by $l[y]$ (13), H^{2n} by H^{2n+1}, and formulas (8) by (14).

Theorem 6. For the existence of separated self-adjoint boundary conditions for the operation $l[y]$ (13) of order $2n+1$ in the cases $\dim H < \infty$ or $\dim H = \infty$, but $n = 0$, it is necessary and sufficient that $\dim H_0^+ = \dim H_0^- \leq \infty$. In this case all such boundary conditions are representable (for $t = 0$) in the form (12), where $\hat{A}_0$ is a self-adjoint operator in $H^{n+} = H^n \oplus H_0^+$,

$$\hat{y}_0 = \{(q_+^{1/2}(0) + V_0 q_-^{1/2}(0)) y_0^{(n)},\ y_0,\ y_0', \dots,\ y_0^{(n-1)}\},$$

$$\hat{y}_0' = \{i(V_0 q_-^{1/2}(0) - q_+^{1/2}(0)) y_0^{(n)},\ y_0^{[2n]},\ y_0^{[2n-1]}, \dots,\ y_0^{[n+1]}\}, \tag{15}$$

V_0 is an arbitrarily fixed isometric operator mapping H_0^- onto H_0^+. If, however, $\dim H = \infty$ and $n \geq 1$, then separated self-adjoint conditions always exist and have the same form (12), but now one must, generally speaking, take

$$\hat{y}_0 = \{y_0^+ + V_1 y_0^-,\ y_0,\ y_0', \dots,\ y_0^{(n-2)}\} \in H^{n+},$$

$$\hat{y}_0' = \{iV_1 y_0^- - iy_0^+,\ y_0^{[2n]},\ y_0^{[2n-1]}, \dots,\ y_0^{[n+2]}\} \in H^{n+}, \tag{16}$$

where

$$y_0^+ = \{q_+^{1/2}(0) y_0^{(n)},\ \tfrac{1}{2}(y_0^{[n+1]} - iy_0^{(n-1)})\} \in H_0^+ \oplus H,$$

$$y_0^- = \{q_-^{1/2}(0) y_0^{(n)},\ \tfrac{1}{2}(y_0^{[n+1]} + iy_0^{(n-1)})\} \in H_0^- \oplus H,$$

V_1 is an arbitrarily fixed isometric operator mapping $H_0^- \oplus H$ onto $H_0^+ \oplus H$. If, in particular, $\dim H_0^+ = \dim H_0^-$, then here too one may use formulas (15) instead of (16).

Corollary. For odd $\dim H < \infty$, for the operation $l[y]$ (13) of odd order there do not exist separated self-adjoint boundary conditions.

Example 3. For the operation $i(P - P^\perp)\dfrac{d}{dt}y$, where P and $P^\perp$ are orthoprojectors onto $H_1 \subseteq H$ and onto $H_1^\perp$, separated self-adjoint boundary conditions exist if $\dim H_1 = \dim H_1^\perp$, and in this case reduce to the form $Py(0) = V_1 P^\perp y(0)$, $Py(b) = V_2 P^\perp y(b)$, where V_1, V_2 are arbitrary isometric operators mapping $H_1^\perp$ onto H_1.

Example 4. Generalized periodic conditions for the operation $l[y]$ (13) of order $2n+1$:

$$y_b^{[2n-k]} = U_{ky} 0^{[2n-k]}, \qquad y_b^{(k)} = U_{ky} 0^{(k)}, \qquad q_\pm^{1/2}(b) y_b^{(n)}$$

$$= V^\pm q_\pm^{1/2}(0) y_0^{(n)}, \qquad k = 0, 1, \dots, n-1,$$

U_k are unitary operators in H, and $V^\pm$ isometrically map $H_0^\pm$ onto $H_b^\pm$, respectively.

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Note: Figure translations are in progress. See original paper for figures.