ON INDECOMPOSABLE SPACES AND TOPOLOGICALLY SELF-DENSE ULTRAFILTERS

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ON INDECOMPOSABLE SPACES AND TOPOLOGICALLY SELF-DENSE ULTRA-FILTERS

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The basic notion in this note is that of a d-end of a topological space (Definition 1). With its help, in the first part we obtain a generalization of Šoke' s theorem on topologically self-dense ultrafilters (Theorem 1), and in the second –the existence of connected H-closed k-decomposable, $1 \leq k < \aleph_0$, but not $(k+1)$-decomposable spaces (Theorem 7), and the coabsoluteness of an arbitrary self-dense Hausdorff space with a minimal space (Theorem 9). We also note Theorem 2, which makes it possible to obtain minimal relaxations of spaces by means of centered systems, and Corollary 2: a θ-homeomorphism preserves extremal disconnectedness. Finally, in the third part the space of d-ends of a given topological space X is constructed and its relation to the space $\theta(X)$ of all ends of the space X is clarified.

Definition 1. A system of subsets of a topological space X is called ***d*-centered*** if the intersection of any finite family of its elements is dense in X.

A d-centered system of subsets of the space X is called a ***d*-end** of the space X if it is not contained in any distinct d-centered system of subsets of X. We shall denote the set of all d-ends of the space X by dX.

It is easily proved that every d-centered system of subsets of the space X is contained in at least one d-end of the space X. Put

$$d_M = \{A \cap M : \ A \in d\}.$$

Proposition 1. 1°. Let $d \in dX$ and $M \subseteq X$. Then there exist disjoint canonical open subsets U and G in X such that

$$[U \cup G] = X, \quad M \cap U \in d_U,$$

and there exists $A \in d$ for which

$$A \cap M \cap G = \Lambda.$$

2°. Let d and d' be two d-ends of the space X. Then there exist disjoint canonical open subsets U and G in X such that

$$[U \cup G] = X, \quad d_U = d'_U,$$

and there exist $A \in d$ and $A' \in d'$ for which

$$A \cap A' \cap G = \Lambda.$$

I. Šoke ([14]) introduced the notion of a topologically self-dense ultrafilter (this is an ultrafilter in the set of points of the segment $[0, 1]$ with a base of self-dense sets) and, under the assumption of the continuum hypothesis, proved the existence of such an ultrafilter. Below we give three proofs of the existence of ultrafilters with a base of self-dense sets in every self-dense T_0-space, not relying on the continuum hypothesis.

Lemma 1. Let** $p \in \theta(X)$ and $d \in dX$, where X is a nonempty topological space. Then

$$p \wedge d = \{U \cap A : \ U \in p, \ A \in d\}$$

is a base of an ultrafilter in the set X.

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* Such systems were considered in ([3]), p. 121.

** An end of the space X is a maximal centered system of open subsets in X. The space $\theta(X)$ is the set of all ends of the space X, endowed with the topology whose base is formed by the sets

$$O_U = \{p \in \theta(X) : \ p \ni U\}.$$

In this topology $\theta(X)$, as shown by S. Iliadis and S. V. Fomin ([7]), is an extremally disconnected bicompactum.

Put $tX = \min\{sU : U \neq \Lambda \text{ and is open in } X\}$*. From Lemma 1 it easily follows

Theorem 1. *In every nonempty crowded T_0-space X ($tX \neq 1$) there exists an ultrafilter with a base each element of which is a crowded subset of X of cardinality tX, whose closure is a canonical closed set in X.*

Hence, by virtue of the well-known theorem of P. S. Aleksandrov stating that every zero-dimensional perfect compactum (i.e., discontinuum) is homeomorphic to the Cantor perfect set $\mathcal{C}$, we immediately obtain

Corollary 1. *There exists an ultrafilter in $\mathcal{C}$ (and, consequently, in $[0, 1]$) with a base each element of which is a crowded countable set whose closure is homeomorphic to $\mathcal{C}$.*

This is the first half of Słupecki' s theorem (([14]), Theorem 10), proved by him under the assumption of the continuum hypothesis.

Second proof. We shall call a system of subsets of a space X q-centered if the intersection of every nonempty finite family of its elements is a nonempty crowded subset of X. A maximal q-centered system of subsets of X will be called a q-end of the space X. A space Y will be called a q-relaxation of the space X if Y is the set X, endowed with the topology with prebase $\mathcal{T} \cup q$, where $\mathcal{T}$ is the topology of the space X and q is a q-centered system of subsets of X.

Theorem 2. *Let X be a crowded space and let q be a q-end in X. Then the q-relaxation of the space X is a minimal** space in which q is an end.*

Since a minimal T_0-space is*** SI and since every end in a nonempty SI, as shown in ([5]), forms a base of an ultrafilter, it follows from Theorem 2 that

Theorem 3. *Every q-end of a nonempty crowded T_0-space X is a base of an ultrafilter in the set X.*

The third proof is simply a generalization of the second: let X be a nonempty crowded T_0-space; let Y be a minimal relaxation of the space X, whose existence was proved by Hewitt ([13]) and Katětov ([8]); then, as was already noted above, every end in Y is a base of an ultrafilter in the set Y and, consequently, in the set X, and every set from this end is, obviously, crowded in the space X.

II. **Definition 2.** A space Y is called a d-relaxation of the space X if Y is the set X, endowed with the topology with prebase $\mathcal{T} \cup d$, where $\mathcal{T}$ is the topology of the space X and d is a d-centered system of subsets of X.

Theorem 4. *Let Y be a relaxation of the space X. Then the following assertions are equivalent: (1) X and Y have one and the same supply of canonical open sets, or, equivalently, one and the same supply of canonical closed sets. (2) X and Y have one and the same associated semiregular space**. (3) Y is a d-relaxa-*

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* $sX = \min\{|A| : A \text{ is crowded in } X\}$.

** A space Y is called a relaxation of the space X if Y consists of the same points as X, and if the identity mapping of Y onto X is continuous. A space is called minimal if it is crowded, and every one of its proper relaxations contains isolated points.

*** One says that X is SI if X is crowded and if every nonempty crowded subset of X is irresolvable, i.e., does not contain two disjoint dense subsets.

**** The canonical open subsets of the space X form a base of a semiregular topology on the set X. The set X, endowed with this topology, is called the semiregular space associated with X (see ([3]), p. 121).

spaces X. (4) The identity mapping X onto Y is θ-discontinuous*.

From Theorem 4 a number of important properties of d-relaxations follow easily.

Proposition 2. A d-relaxation of a connected space is connected.

Proposition 3. Every d-relaxation of an extremally disconnected space is extremally disconnected. Conversely, if a space has at least one extremally disconnected d-relaxation, then the space itself is extremally disconnected.

Proposition 4. A d-relaxation of an H-closed space is H-closed.

The last proposition follows from the fact that, as was proved by S. V. Fomin ([12]), the θ-continuous image of an H-closed space is H-closed.

Proposition 5. Every proper d-relaxation of an arbitrary topological space is non-semiregular (and hence irregular).

Lemma 2. A θ-homeomorphism between two semiregular spaces is a homeomorphism.

Theorem 5. The following assertions concerning topological spaces X and Y consisting of the same points are equivalent:
(1) X and Y have one and the same stock of canonical open sets.
(2) X and Y have one and the same associated semiregular space.
(3) X and Y are d-relaxations of one and the same (semiregular) space.
(4) X and Y are identically θ-homeomorphic.

Corollary 2. If two spaces are θ-homeomorphic and one of them is extremally disconnected, then the other space is also extremally disconnected.

Theorem 6. Let X be a dense-in-itself T_1-space and $d \in dX$. Then the d-relaxation of the space X is an MI-space**.

At the seminar of P. S. Aleksandrov, B. A. Efimov posed the following problem: do there exist connected Hausdorff k-decomposable*** spaces, $2 \leq k < \aleph_0$, which are not $(k+1)$-decomposable? Its solution is given by

Theorem 7. Every connected H-closed k-decomposable space, $1 \leq k < \aleph_0$, has a connected H-closed k-decomposable relaxation that is not $(k+1)$-decomposable****.

Theorem 8. Every dense-in-itself extremally disconnected H-closed space has an H-closed minimal relaxation*****.

Theorem 9. Every dense-in-itself Hausdorff space is coabsolute****** with some minimal space.

* A mapping $f : X \to Y$ is called θ-continuous in the sense of S. V. Fomin ([12]) if, for every point $x \in X$ and for every neighborhood Ofx of its image, there exists a neighborhood Ox of this point such that $f[Ox] \subseteq [Ofx]$. A one-to-one mapping that is θ-continuous in both directions is called a θ-homeomorphism.

** A space is called an MI-space if it is dense in itself and if every subset dense in it is open.

*** A space is k-decomposable if it contains k pairwise disjoint subsets dense in it ($k \geq 1$).

**** Hewitt' s problem ([13])—whether there exist connected Hausdorff indecomposable spaces—was solved by Padmavally ([9]) by the construction of an example. Then Bourbaki (([3]), p.181) and Anderson ([2]) gave a proof of the existence of a connected indecomposable relaxation for every connected Hausdorff space. The existence of k-decomposable Hausdorff spaces, $2 \leq k < \aleph_0$, which are not $(k+1)$-decomposable, was proved in ([4]).

***** Dense-in-itself extremally disconnected H-closed spaces exist: for every dense-in-itself Hausdorff space X, such a space is $\theta(X)$. The existence of H-closed minimal spaces was proved by Katětov ([8]).

****** Two Hausdorff spaces are coabsolute if they have one and the same absolute, i.e. the maximal irreducible θ-continuous preimage. For the first time, the (spectral) theory of absolutes of paracompact Hausdorff spaces was constructed by V. I. Ponomarev ([10]). Then, by the method of centered systems, S. Iliadis ([6]) and, by the spectral method, V. I. Ponomarev ([11]) constructed it already for arbitrary Hausdorff spaces.

III. **Definition 3.** Let X be an arbitrary topological space. The space dX is the set dX, endowed with the topology whose base is formed by the sets

$$O_A = \{d \in dX : \ d \supset A\},$$

where $A \subset X$.

Proposition 6. For every space X, the space dX is zero-dimensional and Hausdorff (and hence completely regular).

Theorem 10. Let $\{U_\lambda : \lambda \in \mathcal{L}\}$ be a family of pairwise disjoint open subsets of the space X, whose union is dense in X. Then dX is homeomorphic to the product

$$\prod \{dU_\lambda : \lambda \in \mathcal{L}\},$$

endowed with the box topology.

Denote by $R(X)$ the maximal open decomposable subspace of the space X (such exists).

Corollary 3. For every space X, the space dX is homeomorphic to $dR(X)$.

Proposition 7. If X is a nonempty decomposable Hausdorff space, then dX is not bicompact.

The construction, due to P. S. Aleksandrov ([1]), of the Stone–Čech extension of an arbitrary completely regular space as applied to a discrete space T gives the following: βT is the set of all ultrafilters in T with the topology whose base is formed by the sets

$$O_E = \{\mathfrak{F} \in \beta T : \ \mathfrak{F} \supset E\},$$

where $E \subset T$. Thus, Lemma 1 defines a natural mapping

$$\mu : \theta(X) \times dX \to \beta T,$$

where T is the set X, endowed with the discrete topology. Namely: $\mu(p, d)$ is the ultrafilter with base $p \wedge d$.

Theorem 11. The mapping μ is continuous and open, and for each $d \in dX$ the mapping μ on the set $\theta(X) \times \{d\}$ is a homeomorphism.

Corollary 4 (B. A. Efimov)*. Every regular extremally disconnected space X is topologically contained in βT, where T is a discrete space of cardinality

$$s\beta X = s\theta(X) \leq sX.$$

Corollary 5. Every infinite extremally disconnected bicompactum (in particular βN, and in general βT, where T is an infinite discrete space) contains an indecomposable space**.

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REFERENCES

1. P. S. Aleksandrov, *Matem. sborn.*, **5** (47), No. 2, 403 (1939).
2. D. R. Anderson, *Proc. Am. Math. Soc.*, **16**, No. 3, 463 (1965).
3. N. Bourbaki, *General Topology, Basic Structures*, Moscow, 1968.
4. A. G. El' kin, *Vestn. Mosk. Univ.*, Ser. 1, Math., Mech., No. 4 (1969).
5. A. G. El' kin, ibid., Ser. 1, Math., Mech., No. 5 (1969).
6. S. Iliadis, *DAN*, **149**, No. 1, 22 (1963).
7. S. Iliadis, S. Fomin, *UMN*, **21**, issue 4 (1966).
8. M. Katetov, *Matem. sborn.*, **21** (63), No. 1, 3 (1947).
9. K. R. Padmavally, *Duke Math. J.*, **20**, No. 4, 513 (1953).
10. V. I. Ponomarev, *Matem. sborn.*, **60** (102), issue 1, 89 (1963).
11. V. I. Ponomarev, *DAN*, **149**, No. 1, 26 (1963).

12. S. Fomin, *Ann. Math.*, **44**, No. 3, 471 (1943).

13. E. Hewitt, *Duke Math. J.*, **10**, 309 (1943).

14. G. Choquet, *Bull. Sci. math.*, 2 ser., **92**, No. 1-2, 41 (1968).

* Unpublished.

** A space is indecomposable if it is dense in itself and is not decomposable.

Note: Figure translations are in progress. See original paper for figures.